b. $\Delta x = \frac{1}{5}$, so $x_1 = \frac{1}{5}$, $x_2 = \frac{2}{5}$, $x_3 = \frac{3}{5}$, $x_4 = \frac{4}{5}$, $x_5 = 1$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots f(x_5)\,\Delta x = \left[\left(\tfrac{1}{5}\right)^3 + \left(\tfrac{2}{5}\right)^3 + \cdots + \left(\tfrac{4}{5}\right)^3 + 1\right]\tfrac{1}{5} = \tfrac{225}{625} = 0.36.$$

c. $\Delta x = \frac{1}{10}$, so $x_1 = \frac{1}{10}$, $x_2 = \frac{2}{10}$, $\ldots$, $x_{10} = 1$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{10})\,\Delta x = \left[\left(\tfrac{1}{10}\right)^3 + \left(\tfrac{2}{10}\right)^3 + \cdots + 1\right]\tfrac{1}{10} = \tfrac{3025}{10{,}000} = 0.3025.$$

d. The Riemann sums seem to approach 0.3.

13. $\Delta x = \frac{2-0}{5} = \frac{2}{5}$, so $x_1 = \frac{1}{5}$, $x_2 = \frac{3}{5}$, $x_3 = \frac{5}{5}$, $x_4 = \frac{7}{5}$, $x_5 = \frac{9}{5}$. Thus,

$$A \approx \left\{\left[\left(\tfrac{1}{5}\right)^2 + 1\right] + \left[\left(\tfrac{3}{5}\right)^2 + 1\right] + \left[\left(\tfrac{5}{5}\right)^2 + 1\right] + \left[\left(\tfrac{7}{5}\right)^2 + 1\right] + \left[\left(\tfrac{9}{5}\right)^2 + 1\right]\right\}\left(\tfrac{2}{5}\right) = \tfrac{580}{125} = 4.64.$$

14. $\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$, so $x_1 = -1$, $x_2 = -\frac{1}{2}$, $x_3 = 0$, $x_4 = \frac{1}{2}$, $x_5 = 1$, $x_6 = \frac{3}{2}$. Thus,

$$A \approx \left\{\left[4-(-1)^2\right] + \left[4-\left(\tfrac{1}{2}\right)^2\right] + \left[4-0^2\right] + \left[4-\left(\tfrac{1}{2}\right)^2\right] + \left[4-1^2\right] + \left[4-\left(\tfrac{3}{2}\right)^2\right]\right\}\left(\tfrac{1}{2}\right) = \tfrac{77}{8} = 9.625.$$

15. $\Delta x = \frac{3-1}{4} = \frac{1}{2}$, so $x_1 = \frac{3}{2}$, $x_2 = \frac{4}{2} = 2$, $x_3 = \frac{5}{2}$, $x_4 = 3$. Thus, $A \approx \left(\frac{1}{3/2} + \frac{1}{2} + \frac{1}{5/2} + \frac{1}{3}\right)\frac{1}{2} \approx 0.95$.

16. $\Delta x = \frac{3-0}{5} = \frac{3}{5}$, so $x_1 = \frac{3}{10}$, $x_2 = \frac{9}{10}$, $x_3 = \frac{15}{10} = \frac{3}{2}$, $x_4 = \frac{21}{10}$, $x_5 = \dfrac{27}{10}$. Thus,

$$A \approx \left(e^{3/10} + e^{9/10} + e^{3/2} + e^{21/10} + e^{27/10}\right)\left(\tfrac{3}{5}\right) \approx 18.8.$$

17. $A \approx 20\left[f(10) + f(30) + f(50) + f(70) + f(90)\right] = 20(80 + 100 + 110 + 100 + 80) = 9400\ \text{ft}^2$.

18. $A \approx 20\left[f(10) + f(30) + f(50) + f(70)\right] = 20(100 + 75 + 80 + 82.5) = 6750\ \text{ft}^2$.

19. False. Take $f(x) = x$, $a = -1$, and $b = 2$. Then $\int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{2} = \frac{1}{2}(4-1) = \frac{3}{2} > 0$, but $f(-1) = -1 < 0$.

20. True. Suppose $f(c) \neq 0$, where $a \le c \le b$. If $c = a$ or $c = b$, then there is an interval $[a, a+s)$ or $(b-t, b]$ where $\left[f(x)\right]^2 > 0$; if $a < c < b$, then there is an interval $(c-u, c+u)$ in (a, b) where $\left[f(x)\right]^2 > 0$. In any case, $\int_I \left[f(x)\right]^2 dx > 0$ on each of these subintervals of $[a, b]$.

6.4 The Fundamental Theorem of Calculus

Concept Questions page 452

1. See the Fundamental Theorem of Calculus on page 444 of the text.

2. See page 448 of the text.

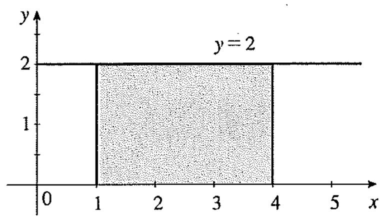
a. It measures the total income generated over $(b-a)$ days.

b. $\int_a^b R(t)\,dt$

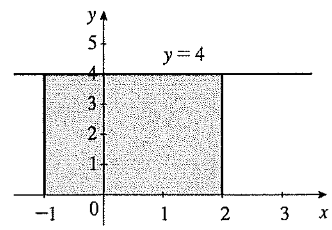
Exercises

page 453

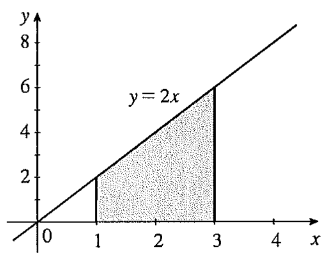
1. $A = \int_1^4 2\,dx = 2x\big|_1^4 = 2\,(4-1) = 6$. The region is a rectangle with area $3 \cdot 2 = 6$.



2. $A = \int_{-1}^2 4\,dx = 4x\big|_{-1}^2 = 8 - (-4) = 12$. The region is a rectangle with area $4\,[2-(-1)] = 12$.

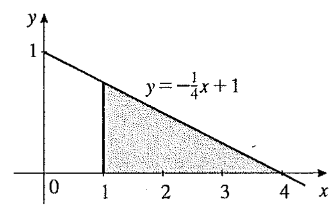


3. $A = \int_1^3 2x\,dx = x^2\big|_1^3 = 9 - 1 = 8$. The region is a parallelogram with area $\frac{1}{2}\,(3-1)\,(2+6) = 8$.



4. $A = \int_1^4 \left(-\frac{1}{4}x + 1\right) dx = \left(-\frac{1}{8}x^2 + x\right)\Big|_1^4$
$= (-2+4) - \left(-\frac{1}{8} + 1\right) = \frac{9}{8}$.

The region is a triangle with area $\frac{1}{2}\,(3)\left(\frac{3}{4}\right) = \frac{9}{8}$.



5. $A = \int_{-1}^2 (2x+3)\,dx = \left(x^2 + 3x\right)\big|_{-1}^2 = (4+6) - (1-3) = 12$.

6. $A = \int_2^4 (4x-1)\,dx = \left(2x^2 - x\right)\big|_2^4 = (32-4) - (8-2) = 22$.

7. $A = \int_{-1}^2 \left(-x^2+4\right) dx = \left(-\frac{1}{3}x^3 + 4x\right)\Big|_{-1}^2 = \left(-\frac{8}{3} + 8\right) - \left(\frac{1}{3} - 4\right) = 9$.

8. $A = \int_0^4 \left(4x - x^2\right) dx = \left(2x^2 - \frac{1}{3}x^3\right)\Big|_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$.

9. $A = \displaystyle\int_1^2 \frac{1}{x}\,dx = \ln|x|\big|_1^2 = \ln 2 - \ln 1 = \ln 2$.

10. $A = \displaystyle\int_2^4 \frac{1}{x^2}\,dx = \int_2^4 x^{-2}\,dx = -\frac{1}{x}\Big|_2^4 = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$.

11. $A = \int_1^9 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^9 = \frac{2}{3}\,(27-1) = \frac{52}{3}$.

12. $A = \int_1^3 x^3\,dx = \frac{1}{4}x^4\Big|_1^3 = \frac{1}{4}\,(81-1) = 20$.

13. $A = \int_{-8}^{-1} \left(1 - x^{1/3}\right) dx = \left(x - \frac{3}{4}x^{4/3}\right)\Big|_{-8}^{-1} = \left(-1 - \frac{3}{4}\right) - (-8-12) = \frac{73}{4}$.

14. $A = \int_1^9 x^{-1/2}\,dx = 2x^{1/2}\Big|_1^9 = 2\,(3-1) = 4.$

15. $A = \int_0^2 e^x\,dx = e^x\big|_0^2 = e^2 - 1 \approx 6.39.$

16. $A = \int_1^2 (e^x - x)\,dx = \left(e^x - \frac{1}{2}x^2\right)\Big|_1^2 = \left(e^2 - 2\right) - \left(e - \frac{1}{2}\right) = e^2 - e - \frac{3}{2} \approx 3.17.$

17. $\int_2^4 3\,dx = 3x\big|_2^4 = 3\,(4-2) = 6.$

18. $\int_{-1}^2 (-2)\,dx = -2x\big|_{-1}^2 = -4 - 2 = -6.$

19. $\int_1^4 (2x+3)\,dx = \left(x^2 + 3x\right)\big|_1^4 = (16+12) - (1+3) = 24.$

20. $\int_{-1}^0 (4-x)\,dx = \left(4x - \frac{1}{2}x^2\right)\Big|_{-1}^0 = 0 - \left(-4 - \frac{1}{2}\right) = \frac{9}{2}.$

21. $\int_{-1}^3 2x^2\,dx = \frac{2}{3}x^3\Big|_{-1}^3 = \frac{2}{3}\,(27) - \frac{2}{3}\,(-1) = \frac{56}{3}.$

22. $\int_0^2 8x^3\,dx = 2x^4\big|_0^2 = 32.$

23. $\int_{-2}^2 (x^2-1)dx = \left(\frac{1}{3}x^3 - x\right)\Big|_{-2}^2 = \left(\frac{8}{3} - 2\right) - \left(-\frac{8}{3} + 2\right) = \frac{4}{3}.$

24. $\int_1^4 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_1^4 = \frac{2}{3}\,(8) - \frac{2}{3}\,(1) = \frac{14}{3}.$

25. $\int_1^8 2x^{1/3}\,dx = 2\cdot\frac{3}{4}x^{4/3}\Big|_1^8 = \frac{3}{2}\,(16-1) = \frac{45}{2}.$

26. $\int_1^4 2x^{-3/2}\,dx = 2\left(-2x^{-1/2}\right)\big|_1^4 = -4\left(\frac{1}{2} - 1\right) - 2.$

27. $\int_0^1 \left(x^3 - 2x^2 + 1\right)dx = \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + x\right)\Big|_0^1 = \frac{1}{4} - \frac{2}{3} + 1 = \frac{7}{12}.$

28. $\int_1^2 \left(t^5 - t^3 + 1\right)dt = \left(\frac{1}{6}t^6 - \frac{1}{4}t^4 + t\right)\Big|_1^2 = \left[\frac{1}{6}\,(64) - \frac{1}{4}\,(16) + 2\right] - \left(\frac{1}{6} - \frac{1}{4} + 1\right) = \frac{31}{4}.$

29. $\displaystyle\int_1^4 \frac{1}{x}\,dx = \ln|x|\big|_1^4 = \ln 4 - \ln 1 = \ln\frac{4}{1} = \ln 4.$

30. $\displaystyle\int_1^3 \frac{2}{x}\,dx = 2\ln|x|\big|_1^3 = 2\ln 3.$

31. $\int_0^4 x\left(x^2-1\right)dx = \int_0^4 \left(x^3 - x\right)dx = \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Big|_0^4 = 64 - 8 = 56.$

32. $\int_0^2 (x-4)(x-1)\,dx = \int_0^2 \left(x^2 - 5x + 4\right)dx = \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x\right)\Big|_0^2 = \frac{8}{3} - 10 + 8 = \frac{2}{3}.$

33. $\int_1^3 \left(t^2 - t\right)^2 dt = \int_1^3 \left(t^4 - 2t^3 + t^2\right)dt = \left(\frac{1}{5}t^5 - \frac{1}{2}t^4 + \frac{1}{3}t^3\right)\Big|_1^3 = \left(\frac{243}{5} - \frac{81}{2} + \frac{27}{3}\right) - \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) = \frac{512}{30} = \frac{256}{15}.$

34. $\int_{-1}^{1}(x^2-1)^2\,dx=\int_{-1}^{1}(x^4-2x^2+1)\,dx=\left(\frac{1}{5}x^5-\frac{2}{3}x^3+x\right)\Big|_{-1}^{1}=\left(\frac{1}{5}-\frac{2}{3}+1\right)-\left(-\frac{1}{5}+\frac{2}{3}-1\right)=\frac{16}{15}.$

35. $\int_{-3}^{-1}x^{-2}dx=-\frac{1}{x}\Big|_{-3}^{-1}=1-\frac{1}{3}=\frac{2}{3}.$

36. $\int_{1}^{2}2x^{-3}\,dx=-\frac{1}{x^2}\Big|_{1}^{2}=-\frac{1}{4}+1=\frac{3}{4}.$

37. $\int_{1}^{4}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx=\int_{1}^{4}\left(x^{1/2}-x^{-1/2}\right)dx=\left(\frac{2}{3}x^{3/2}-2x^{1/2}\right)\Big|_{1}^{4}=\left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-2\right)=\frac{8}{3}.$

38. $\int_{0}^{1}\sqrt{2x}\left(\sqrt{x}+\sqrt{2}\right)dx=\int_{0}^{1}\left(\sqrt{2}x+2\sqrt{x}\right)dx=\left(\frac{\sqrt{2}}{2}x^2+\frac{4}{3}x^{3/2}\right)\Big|_{0}^{1}=\frac{\sqrt{2}}{2}+\frac{4}{3}.$

39. $\int_{1}^{4}\frac{3x^3-2x^2+4}{x^2}\,dx=\int_{1}^{4}\left(3x-2+4x^{-2}\right)dx=\left(\frac{3}{2}x^2-2x-\frac{4}{x}\right)\Big|_{1}^{4}$
$=\left(24-8-1\right)-\left(\frac{3}{2}-2-40\right)=\frac{39}{2}.$

40. $\int_{1}^{2}\left(1+\frac{1}{u}+\frac{1}{u^2}\right)du=\left(u+\ln u-\frac{1}{u}\right)\Big|_{1}^{2}=\left(2+\ln 2-\frac{1}{2}\right)-(1+0-1)=\frac{3}{2}+\ln 2.$

41. a. The change in the number of annual personal bankruptcy filings between September 30, 2010 and September 2012 was $N(t)=\int_0^2[-R(t)]\,dt=-\int_0^2(0.077t+0.0825)\,dt=\left[-0.0385t^2+0.0825t\right]_0^2\approx-0.319$, a decline of approximately 319,000.

b. The approximate number of personal bankruptcy filings in 2012 was $N(2)=N(0)+\int_0^2N'(t)\,dt=1.538-0.319=1.219$, or 1,219,000.

42. Projected spending in 2016 is $A(6)=A(0)+\int_0^6R(t)\,dt=317+\int_0^6(1.0952t+17.357)\,dt=317+\left[0.5476t^2+17.357t\right]_0^6\approx440.856$, or approximately \$440.9 million.

43. a. $C(300)-C(0)=\int_0^{300}\left(0.0003x^2-0.12x+20\right)dx=\left(0.0001x^3-0.06x^2+20x\right)\big|_0^{300}$
$=0.0001(300)^3-0.06(300)^2+20(300)=3300.$
Therefore, $C(300)=3300+C(0)=3300+800=\$4100.$

b. $\int_{200}^{300}C'(x)\,dx=\left(0.0001x^3-0.06x^2+20x\right)\big|_{200}^{300}$
$=\left[0.0001(300)^3-0.06(300)^2+20(300)\right]-\left[0.0001(200)^3-0.06(200)^2+20(200)\right]=\$900.$

44. a. $R(200)=\int_0^{200}(-0.1x+40)\,dx=\left(-0.05x^2+40x\right)\big|_0^{200}=6000$, or \$6000.

b. $R(300)-R(200)=\int_{200}^{300}(-0.1x+40)dx=\left(-0.05x^2+40x\right)\big|_{200}^{300}=7500-6000=1500$, or \$1500.

45. a. The profit is
$\int_0^{200}\left(-0.0003x^2+0.02x+20\right)dx+P(0)=\left(-0.0001x^3+0.01x^2+20x\right)\big|_0^{200}+P(0)$
$=3600+P(0)=3600-800$, or \$2800.

b. $\int_{200}^{220}P'(x)\,dx=P(220)-P(200)=\left(-0.0001x^3+0.01x^2+20x\right)\big|_{200}^{220}=219.20$, or \$219.20.