66. Let $u = 1 + 2.449e^{-0.3277t}$, so $du = -0.802537e^{-0.3277t}\,dt$ and $e^{-0.3277t}\,dt = -1.24605\,du$. Then

$$h(t) = \int \frac{52.8706e^{-0.3277t}}{\left(1 + 2.449e^{-0.3277t}\right)^2}\,dt = 52.8706\,(-1.24605)\int \frac{du}{u^2} = 65.8794u^{-1} + C = \frac{65.8794}{1 + 2.449e^{-0.3277t}} + C.$$

$h(0) = \dfrac{65.8794}{1 + 2.449} + C = 19.4$, so $h(t) = \dfrac{65.8794}{1 + 2.449e^{-0.3277t}} + 0.3$ and hence

$h(8) = \dfrac{65.8794}{1 + 2.449e^{-0.3277(8)}} + 0.3 \approx 56.22$, or 56.22 inches.

67. $A(t) = \int A'(t)\,dt = r\int e^{-at}\,dt$. Let $u = -at$, so $du = -a\,dt$ and $dt = -\frac{1}{a}\,du$. Then $A(t) = r\left(-\frac{1}{a}\right)\int e^u\,du = -\frac{r}{a}e^u + C = -\frac{r}{a}e^{-at} + C$. $A(0) = 0$ implies $-\frac{r}{a} + C = 0$, so $C = \frac{r}{a}$. Therefore, $A(t) = -\frac{r}{a}e^{-at} + \frac{r}{a} = \frac{r}{a}\left(1 - e^{-at}\right)$.

68. $x(t) = \int x'(t)\,dt = \dfrac{1}{V}(ac - bx_0)\int e^{-bt/V}\,dt$. Let $u = -\dfrac{bt}{V}$, so $du = -\dfrac{b}{V}\,dt$ and $dt = -\dfrac{V}{b}\,du$.

Then $x(t) = \dfrac{1}{V}(ac - bx_0)\displaystyle\int\left(-\frac{V}{b}e^u\right)du = \left(-\frac{ac}{b} + x_0\right)e^u = \left(-\frac{ac}{b} + x_0\right)e^{-bt/V} + C$. Because $x(0) = \left(-\dfrac{ac}{b} + x_0\right) + C = x_0$, we have $C = \dfrac{ac}{b}$, and hence $x(t) = \dfrac{ac}{b} + \left(x_0 - \dfrac{ac}{b}\right)e^{-bt/V}$

69. True. Let $I = \int xf\left(x^2\right)dx$ and put $u = x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so $I = \frac{1}{2}\int f(u)\,du = \frac{1}{2}\int f(x)\,dx$.

70. False. Let $f(x) = x$, $a = 2$, and $b = 3$. Then $\int f(ax + b)\,dx = \int (2x + 3)\,dx = x^2 + 3x + C_1$. On the other hand $a\int f(x)\,dx = 2\int x\,dx = x^2 + C_2 \neq \int f(ax + b)\,dx$.

71. True. Put $u = kx$. Then $du = k\,dx$ and $dx = (1/k)\,du$. Thus, $I = \int e^{kx} f\left(e^{kx}\right)dx = \int e^u f\left(e^u\right)(1/k)\,du = \frac{1}{k}\int e^u f\left(e^u\right)du$. Next, we put $w = e^u$, so $dw = e^u du$. Then $I = \frac{1}{k}\int f(w)\,dw = \frac{1}{k}\int f(x)\,dx$.

72. True. Put $u = \ln x$, so $du = \dfrac{dx}{x}$. Then $\displaystyle\int \frac{f(\ln x)}{x}\,dx = \int f(u)\,du = \int f(x)\,dx$.

6.3 Area and the Definite Integral

Concept Questions page 442

1. See page 438 in the text.

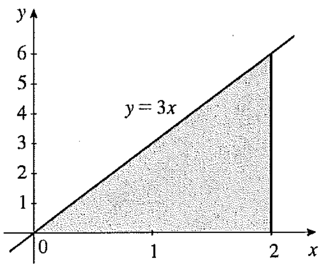
2. See pages 440–441 in the text.

Exercises page 442

1. $\frac{1}{3}(1.9 + 1.5 + 1.8 + 2.4 + 2.7 + 2.5) = \frac{12.8}{3} \approx 4.27$.

2. $\frac{1}{4}(4.5 + 8.0 + 8.5 + 6.0 + 4.0 + 3.0 + 2.5 + 2.0) = \frac{38.5}{4} = 9.625$.

3. a.



$A = \frac{1}{2}(2)(6) = 6.$

b. $\Delta x = \frac{2}{4} = \frac{1}{2}$, so $x_1 = 0, x_2 = \frac{1}{2}, x_3 = 1, x_4 = \frac{3}{2}$. Thus,
$A \approx \frac{1}{2}\left[3(0) + 3\left(\frac{1}{2}\right) + 3(1) + 3\left(\frac{3}{2}\right)\right] = \frac{9}{2} = 4.5.$

c. $\Delta x = \frac{2}{8} = \frac{1}{4}$, so $x_1 = 0, \ldots, x_8 = \frac{7}{4}$. Thus,

$$A \approx \frac{1}{4}\left[3(0) + 3\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right) + 3\left(\frac{3}{4}\right) + 3(1) + 3\left(\frac{5}{4}\right) + 3\left(\frac{3}{2}\right) + 3\left(\frac{7}{4}\right)\right] = \frac{21}{4} = 5.25.$$

d. Yes.

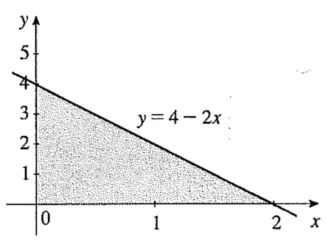
4. a. $A = 6$. See the graph in the solution to Exercise 3.

b. $x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2$. Thus, $A \approx \frac{1}{2}\left[3\left(\frac{1}{2}\right) + 3(1) + 3\left(\frac{3}{2}\right) + 3(2)\right] = 7.5.$

c. $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, \ldots, x_8 = 2$. Thus, $A \approx \frac{1}{4}\left[3\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right) + 3\left(\frac{3}{4}\right) + \cdots + 3\left(\frac{7}{4}\right) + 3(2)\right] = 6.75.$

d. Yes.

5. a.



$A = 4$

b. $\Delta x = \frac{2}{5} = 0.4$, so $x_1 = 0, x_2 = 0.4, x_3 = 0.8, x_4 = 1.2,$
$x_5 = 1.6$. Thus,

$$A \approx 0.4\{[4 - 2(0)] + [4 - 2(0.4)] + [4 - 2(0.8)] + [4 - 2(1.2)] + [4 - 2(1.6)]\} = 4.8.$$

c. $\Delta x = \frac{2}{10} = 0.2$, so $x_1 = 0, x_2 = 0.2, x_3 = 0.4, \ldots, x_{10} = 1.8$.
Thus,

$$A \approx 0.2\{[4 - 2(0)] + [4 - 2(0.2)] + [4 - 2(0.4)] + [4 - 2(0.6)] + [4 - 2(0.8)] + [4 - 2(1.0)] + [4 - 2(1.2)] + [4 - 2(1.4)] + [4 - 2(1.6)] + [4 - 2(1.8)]\} = 4.4.$$

d. Yes.

6. a. $A = 4$. See the graph in the solution to Exercise 5.

b. $\Delta x = 0.4$, so $x_1 = 0.4, x_2 = 0.8, x_3 = 1.2, x_4 = 1.6, x_5 = 2$. Thus,
$A \approx 0.4\{[4 - 2(0.4)] + [4 - 2(0.8)] + \cdots + [4 - 2(2)]\} \approx 3.2.$

c. $\Delta x = 0.2$, so $x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, \ldots, x_{10} = 2$. Thus,
$A \approx 0.2\{[4 - 2(0.2)] + [4 - 2(0.4)] + \cdots + [4 - 2(2)]\} \approx 3.6.$

d. Yes.

7. a. $\Delta x = \frac{4-2}{2} = 1$, so $x_1 = 2.5, x_2 = 3.5$. The Riemann sum is $1\left(2.5^2 + 3.5^2\right) = 18.5.$

b. $\Delta x = \frac{4-2}{5} = 0.4$, so $x_1 = 2.2, x_2 = 2.6, x_3 = 3.0, x_4 = 3.4, x_5 = 3.8$. The Riemann sum is
$0.4\left(2.2^2 + 2.6^2 + 3.0^2 + 3.4^2 + 3.8^2\right) = 18.64.$

c. $\Delta x = \frac{4-2}{10} = 0.2$, so $x_1 = 2.1, x_2 = 2.3, x_2 = 2.5, \ldots, x_{10} = 3.9$. The Riemann sum is
$0.2\left(2.1^2 + 2.3^2 + 2.5^2 + 2.7^2 + 2.9^2 + 3.1^2 + 3.3^2 + 3.5^2 + 3.7^2 + 3.9^2\right) = 18.66.$

d. The area appears to be $18\frac{2}{3}$.

8. a. $\Delta x = \frac{4-2}{2} = 1$, so $x_1 = 2, x_2 = 3$. The Riemann sum is $1\left(2^2 + 3^2\right) = 13$.

b. $\Delta x = \frac{4-2}{5} = 0.4$, so $x_1 = 2, x_2 = 2.4, x_3 = 2.8, x_4 = 3.2, x_5 = 3.6$. The Riemann sum is $0.4\left(2^2 + 2.4^2 + 2.8^2 + 3.2^2 + 3.6^2\right) = 16.32$.

c. $\Delta x = \frac{4-2}{10} = 0.2$, so $x_1 = 2, x_2 = 2.2, x_3 = 2.4, \ldots, x_{10} = 3.8$. The Riemann sum is $0.2\left(2^2 + 2.2^2 + 2.4^2 + \cdots + 3.8^2\right) = 17.48$.

d. The area appears to be $17\frac{1}{2}$.

9. a. $\Delta x = \frac{4-2}{2} = 1$, so $x_1 = 3, x_2 = 4$. The Riemann sum is $(1)\left(3^2 + 4^2\right) = 25$.

b. $\Delta x = \frac{4-2}{5} = 0.4$, so $x_1 = 2.4, x_2 = 2.8, x_3 = 3.2, x_4 = 3.6, x_5 = 4$. The Riemann sum is $0.4\left(2.4^2 + 2.8^2 + \cdots + 4^2\right) = 21.12$.

c. $\Delta x = \frac{4-2}{10} = 0.2$, so $x_1 = 2.2, x_2 = 2.4, x_3 = 2.6, \ldots, x_{10} = 4$. The Riemann sum is $0.2\left(2.2^2 + 2.4^2 + 2.6^2 + 2.8^2 + 3.0^2 + 3.2^2 + 3.4^2 + 3.6^2 + 3.8^2 + 4^2\right) = 19.88$.

d. The area appears to be 19.9.

10. a. $\Delta x = \frac{1-0}{2} = \frac{1}{2}$, so $x_1 = \frac{1}{4}, x_2 = \frac{3}{4}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x = \left[\left(\tfrac{1}{4}\right)^3 + \left(\tfrac{3}{4}\right)^3\right]\tfrac{1}{2} = \left(\tfrac{1}{64} + \tfrac{27}{64}\right)\tfrac{1}{2} = \tfrac{7}{32} = 0.21875.$$

b. $\Delta x = \frac{1-0}{5} = \frac{1}{5}$, so $x_1 = \frac{1}{10}, x_2 = \frac{3}{10}, x_3 = \frac{5}{10}, x_4 = \frac{7}{10}, x_5 = \frac{9}{10}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_5)\,\Delta x = \left[\left(\tfrac{1}{10}\right)^3 + \left(\tfrac{3}{10}\right)^3 + \cdots + \left(\tfrac{9}{10}\right)^3\right]\tfrac{1}{5} = \tfrac{1}{5000}(1 + 27 + \cdots + 729)$$
$$= \tfrac{1225}{5000} = 0.245.$$

c. $\Delta x = \frac{1-0}{10} = \frac{1}{10}$, so $x_1 = \frac{1}{20}, x_2 = \frac{3}{20}, x_3 = \frac{5}{20}, \ldots, x_{10} = \frac{19}{20}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{10})\,\Delta x = \left[\left(\tfrac{1}{20}\right)^3 + \left(\tfrac{3}{20}\right)^3 + \cdots + \left(\tfrac{19}{20}\right)^3\right]\tfrac{1}{10} = \tfrac{19{,}900}{80{,}000} \approx 0.24875.$$

d. The Riemann sums seem to approach $\frac{1}{4}$.

11. a. $\Delta x = \frac{1}{2}$, so $x_1 = 0, x_2 = \frac{1}{2}$. The Riemann sum is $f(x_1)\,\Delta x + f(x_2)\,\Delta x = \left[(0)^3 + \left(\frac{1}{2}\right)^3\right]\frac{1}{2} = \frac{1}{16} = 0.0625$.

b. $\Delta x = \frac{1}{5}$, so $x_1 = 0, x_2 = \frac{1}{5}, x_3 = \frac{2}{5}, x_4 = \frac{3}{5}, x_5 = \frac{4}{5}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_5)\,\Delta x = \left[\left(\tfrac{1}{5}\right)^3 + \left(\tfrac{2}{5}\right)^3 + \cdots + \left(\tfrac{4}{5}\right)^3\right]\tfrac{1}{5} = \tfrac{100}{625} = 0.16.$$

c. $\Delta x = \frac{1}{10}$, so $x_1 = 0, x_2 = \frac{1}{10}, x_3 = \frac{2}{10}, \ldots, x_{10} = \frac{9}{10}$. The Riemann sum is

$$f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{10})\,\Delta x$$
$$= \left[0^3 + \left(\tfrac{1}{10}\right)^3 + \left(\tfrac{2}{10}\right)^3 + \left(\tfrac{3}{10}\right)^3 + \left(\tfrac{4}{10}\right)^3 + \left(\tfrac{5}{10}\right)^3 + \left(\tfrac{6}{10}\right)^3 + \left(\tfrac{7}{10}\right)^3 + \left(\tfrac{8}{10}\right)^3 + \left(\tfrac{9}{10}\right)^3\right]\tfrac{1}{10}$$
$$= \tfrac{2025}{10{,}000} = 0.2025 \approx 0.2.$$

d. The Riemann sums seem to approach 0.2.

12. a. $\Delta x = \frac{1}{2}$, so $x_1 = \frac{1}{2}, x_2 = 1$. The Riemann sum is $f(x_1)\,\Delta x + f(x_2)\,\Delta x = \left[\left(\frac{1}{2}\right)^3 + 1^3\right]\frac{1}{2} = 0.5625$.