4.4 Optimization I

Concept Questions

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1. **a.** A function f has an absolute maximum at a if $f(x) \leq f(a)$ for all x in the domain of f.

   b. A function f has an absolute minimum at a if $f(x) \geq f(a)$ for all x in the domain of f.

2. See the procedure given on page 307 of the text.

Exercises

page 313

1. f has no absolute extremum.

2. f has an absolute minimum at $\left(-2, -\frac{1}{2}\right)$ and an absolute maximum at $\left(2, \frac{1}{2}\right)$.

3. f has an absolute minimum at $(0, 0)$.

4. f has an absolute minimum at $(0, 0)$ and no absolute maximum.

5. f has an absolute minimum at $(0, -2)$ and an absolute maximum at $(1, 3)$.

6. f has no absolute extremum.

7. f has an absolute minimum at $\left(\frac{3}{2}, -\frac{27}{16}\right)$ and an absolute maximum at $(-1, 3)$.

8. f has an absolute minimum at $(0, -3)$ and an absolute maximum at $(3, 1)$.

9. The graph of $f(x) = 2x^2 + 3x - 4$ is a parabola that opens upward. Therefore, the vertex of the parabola is the absolute minimum of f. To find the vertex, we solve the equation $f'(x) = 4x + 3 = 0$, finding $x = -\frac{3}{4}$. We conclude that the absolute minimum value is $f\left(-\frac{3}{4}\right) = -\frac{41}{8}$.

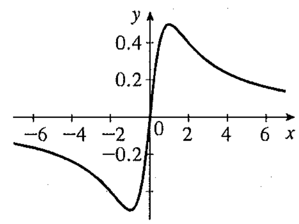
10. The graph of $g(x) = -x^2 + 4x + 3$ is a parabola that opens downward. Therefore, the vertex of the parabola is the absolute maximum of f. To find the vertex, we solve the equation $g'(x) = -2x + 4 = 0$, finding $x = 2$. We conclude that the absolute maximum value is $f(2) = 7$.

11. Because $\lim_{x \to -\infty} x^{1/3} = -\infty$ and $\lim_{x \to \infty} x^{1/3} = \infty$, we see that h is unbounded. Therefore, it has no absolute extremum.

12. From the graph of f (see Figure 15(b) on page 259 of the text), we see that $(0, 0)$ is an absolute minimum of f. There is no absolute maximum because $\lim_{x \to \infty} x^{2/3} = \infty$.

13. $f(x) = \dfrac{1}{1 + x^2}$. Using the techniques of graphing, we sketch the graph of f (see Figure 40 on page 278 of the text). The absolute maximum of f is $f(0) = 1$. Alternatively, observe that $1 + x^2 \geq 1$ for all real values of x. Therefore, $f(x) \leq 1$ for all x, and we see that the absolute maximum is attained when $x = 0$.

14. $f(x) = \frac{x}{1+x^2}$. Because f is defined for all x in $(-\infty, \infty)$, we use the graphical method. Using the techniques of graphing, we sketch the graph of f. From the graph we see that f has an absolute maximum at $\left(1, \frac{1}{2}\right)$ and an absolute minimum at $\left(-1, -\frac{1}{2}\right)$.



15. $f(x) = x^2 - 2x - 3$ and $f'(x) = 2x - 2 = 0$, so $x = 1$ is a critical number. From the table, we conclude that the absolute maximum value is $f(-2) = 5$ and the absolute minimum value is $f(1) = -4$.

x	-2	1	3
$f(x)$	5	-4	0

16. $g(x) = x^2 - 2x - 3$, so $g'(x) = 2x - 2 = 0$ implies that $x = 1$ is a critical number. From the table, we conclude that g has an absolute minimum at $(1, -4)$ and an absolute maximum at $(4, 5)$.

x	0	1	4
$f(x)$	-3	-4	5

17. $f(x) = -x^2 + 4x + 6$; The function f is continuous and defined on the closed interval $[0, 5]$. $f'(x) = -2x + 4$, and so $x = 2$ is a critical number. From the table, we conclude that $f(2) = 10$ is the absolute maximum value and $f(5) = 1$ is the absolute minimum value.

x	0	2	5
$f(x)$	6	10	1

18. $f(x) = -x^2 + 4x + 6$; The function f is continuous and defined on the closed interval $[3, 6]$. $f'(x) = -2x + 4$, so $x = 2$ is a critical number. But this point lies outside the given interval. From the table, we conclude that $f(3) = 9$ is the absolute maximum value and $f(6) = -6$ is the absolute minimum value.

x	3	6
$f(x)$	9	-6

19. The function $f(x) = x^3 + 3x^2 - 1$ is continuous and defined on the closed interval $[-3, 2]$ and differentiable in $(-3, 2)$. The critical numbers of f are found by solving $f'(x) = 3x^2 + 6x = 3x(x+2) = 0$, giving $x = -2$ and $x = 0$. From the table, we see that the absolute maximum value of f is $f(2) = 19$ and the absolute minimum value is $f(-3) = f(0) = -1$.

x	-3	-2	0	2
$f(x)$	-1	3	-1	19

20. The function $g(x) = x^3 + 3x^2 - 1$ is continuous on the closed interval $[-3, 1]$ and differentiable in $(-3, 1)$. The critical numbers of g are found by solving $g'(x) = 3x^2 + 6x = 3x(x+2) = 0$, giving $x = -2$ and $x = 0$. From the table we see that the absolute maximum value of g is $g(1) = g(-2) = 3$ and the absolute minimum value of g is $g(-3) = g(0) = -1$.

x	-3	-2	0	1
$g(x)$	-1	3	-1	3

21. The function $g(x) = 3x^4 + 4x^3$ is continuous on the closed interval $[-2, 1]$ and differentiable in $(-2, 1)$. The critical numbers of g are found by solving $g'(x) = 12x^3 + 12x^2 = 12x^2(x+1) = 0$, giving $x = 0$ and $x = -1$. From the table, we see that $g(-2) = 16$ is the absolute maximum value of g and $g(-1) = -1$ is the absolute minimum value of g.

x	-2	-1	0	1
$g(x)$	16	-1	0	7

22. $f(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$ is continuous on the closed interval $[-2, 3]$ and differentiable in the open interval $(-2, 3)$. The critical numbers of f are found by solving $f'(x) = 2x^3 - 2x^2 - 4x = 2x\left(x^2 - x - 2\right) = 2x(x-2)(x+1) = 0$, giving $x = -1, 0$, and 2 as critical numbers. From the table we see that the absolute maximum value of f is $f(-2) = \frac{25}{3}$ and the absolute minimum value of f is $f(2) = -\frac{7}{3}$.

x	-2	-1	0	2	3
$f(x)$	$\frac{25}{3}$	$\frac{13}{6}$	3	$-\frac{7}{3}$	$\frac{15}{2}$

23. $f(x) = \dfrac{x+1}{x-1}$ on $[2, 4]$. Next, we compute $f'(x) = \dfrac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = -\dfrac{2}{(x-1)^2}$. Because there is no critical number ($x = 1$ is not in the domain of f), we need only test the endpoints. We conclude that $f(4) = \frac{5}{3}$ is the absolute minimum value and $f(2) = 3$ is the absolute maximum value.

24. $g(t) = \dfrac{t}{t-1}$, so $g'(t) = \dfrac{(t-1) - t}{(t-1)^2} = -\dfrac{1}{(t-1)^2}$. Because there is no critical number ($t = 1$ is not in the domain of g), we need only test the endpoints. We conclude that $g(2) = 2$ is the absolute maximum value and $g(4) = \frac{4}{3}$ is the absolute minimum value.

25. $f(x) = 4x + \dfrac{1}{x}$ is continuous on $[1, 4]$ and differentiable in $(1, 4)$. To find the critical numbers of f, we solve $f'(x) = 4 - \dfrac{1}{x^2} = 0$, obtaining $x = \pm\frac{1}{2}$. Because these critical numbers lie outside the interval $[1, 4]$, they are not candidates for the absolute extrema of f. Evaluating f at the endpoints of the interval $[1, 4]$, we find that the absolute maximum value of f is $f(4) = \frac{65}{4}$, and the absolute minimum value of f is $f(1) = 5$.

26. $f(x) = 9x - \dfrac{1}{x}$ is continuous on $[1, 3]$ and differentiable in $(1, 3)$. To find the critical numbers of f, we solve $f'(x) = 9 + \frac{1}{x^2} = 0$, obtaining $x^2 = -\frac{1}{9}$ which has no solution. Evaluating f at the endpoints of the interval $[1, 3]$, we find that the absolute minimum value is $f(1) = 8$ and the absolute maximum value is $f(3) = \frac{80}{3}$.

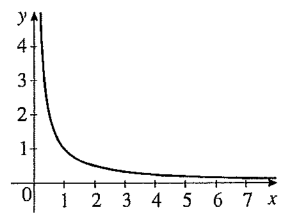
27. $f(x) = \frac{1}{2}x^2 - 2\sqrt{x} = \frac{1}{2}x^2 - 2x^{1/2}$. To find the critical numbers of f, we solve $f'(x) = x - x^{-1/2} = 0$, or $x^{3/2} - 1 = 0$, obtaining $x = 1$. From the table, we conclude that $f(3) \approx 1.04$ is the absolute maximum value and $f(1) = -\frac{3}{2}$ is the absolute minimum value.

x	0	1	3
$f(x)$	0	$-\frac{3}{2}$	$\frac{9}{2} - 2\sqrt{3} \approx 1.04$

28. The function $g(x) = \frac{1}{8}x^2 - 4\sqrt{x} = \frac{1}{8}x^2 - 4x^{1/2}$ is continuous on the closed interval $[0, 9]$ and differentiable in $(0, 9)$. To find the critical numbers of g, we first compute $g'(x) = \frac{1}{4}x - 2x^{-1/2} = \frac{1}{4}x^{-1/2}\left(x^{3/2} - 8\right)$. Setting $g'(x) = 0$, we have $x^{3/2} = 8$, or $x = 4$. From the table, we conclude that $g(4) = -6$ is the absolute minimum value and $g(0) = 0$ is the absolute maximum value of g.

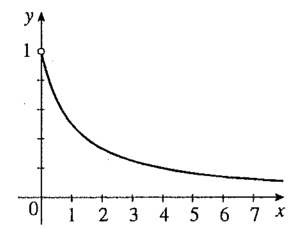
x	0	4	9
$f(x)$	0	-6	$-\frac{15}{8}$

29.



From the graph of $f(x) = \dfrac{1}{x}$ for $x > 0$, we conclude that f has no absolute extremum.

30.



From the graph of $g(x) = \dfrac{1}{x+1}$ for $x > 0$, we conclude that g has no absolute extremum.

31. $f(x) = 3x^{2/3} - 2x$. The function f is continuous on $[0, 3]$ and differentiable on $(0, 3)$. To find the critical numbers of f, we solve $f'(x) = 2x^{-1/3} - 2 = 0$, obtaining $x = 1$ as the critical number. From the table, we conclude that the absolute maximum value is $f(1) = 1$ and the absolute minimum value is $f(0) = 0$.

x	0	1	3
$f(x)$	0	1	$3^{5/3} - 6 \approx 0.24$

32. $g(x) = x^2 + 2x^{2/3}$, so $g'(x) = 2x + \frac{4}{3}x^{-1/3} = \frac{2}{3}x^{-1/3}\left(3x^{4/3} + 2\right)$ is never zero, but $g'(x)$ is not defined at $x = 0$, which is a critical number of g. From the table, we conclude that $g(-2) = g(2) = 4 + 2^{5/3}$ give the absolute maximum value and $g(0) = 0$ gives the absolute minimum value.

x	-2	0	2
$g(x)$	$4 + 2^{5/3}$	0	$4 + 2^{5/3}$

33. $f(x) = x^{2/3}\left(x^2 - 4\right)$, so $f'(x) = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}\left(x^2 - 4\right) = \frac{2}{3}x^{-1/3}\left[3x^2 + \left(x^2 - 4\right)\right] = \dfrac{8\left(x^2 - 1\right)}{3x^{1/3}} = 0$. Observe that f' is not defined at $x = 0$. Furthermore, $f'(x) = 0$ at $x \pm 1$. So the critical numbers of f are -1 and 0, and 1. From the table, we see that f has absolute minima at $(-1, -3)$ and $(1, -3)$ and absolute maxima at $(0, 0)$ and $(2, 0)$.

x	-1	0	1	2
$f(x)$	-3	0	-3	0

34. The function is the same as that of Exercise 33. From the table, we see that f has a absolute minima at $(-1, -3)$ and $(1, -3)$ and an absolute maximum at $\left(3, 5 \cdot 3^{2/3}\right)$.

x	-1	0	1	3
$f(x)$	-3	0	-3	$5 \cdot 3^{2/3}$

35. $f(x) = \dfrac{x}{x^2+2}$. To find the critical numbers of f, we solve $f'(x) = \dfrac{(x^2+2) - x(2x)}{(x^2+2)^2} = \dfrac{2-x^2}{(x^2+2)^2} = 0$, obtaining $x = \pm\sqrt{2}$. Because $x = -\sqrt{2}$ lies outside $[-1, 2]$, $x = \sqrt{2}$ is the only critical number in the given interval.

From the table, we conclude that $f\left(\sqrt{2}\right) = \frac{\sqrt{2}}{4} \approx 0.35$ is the absolute maximum value and $f(-1) = -\frac{1}{3}$ is the absolute minimum value.

x	-1	$\sqrt{2}$	2
$f(x)$	$-\frac{1}{3}$	$\frac{\sqrt{2}}{4} \approx 0.35$	$\frac{1}{3}$

36. $f'(x) = \dfrac{d}{dx}\left(x^2+2x+5\right)^{-1} = -\left(x^2+2x+5\right)^{-2}(2x+2) = \dfrac{-2(x+1)}{\left(x^2+2x+5\right)^2}$. Setting $f'(x) = 0$ gives $x = -1$ as a critical number. From the table, we see that f has an absolute minimum at $\left(1, \frac{1}{8}\right)$ and an absolute maximum at $\left(-1, \frac{1}{4}\right)$.

x	-2	-1	1
$f(x)$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{8}$

37. The function $f(x) = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x}{\left(x^2+1\right)^{1/2}}$ is continuous on the closed interval $[-1, 1]$ and differentiable on $(-1, 1)$. To find the critical numbers of f, we first compute

$$f'(x) = \frac{\left(x^2+1\right)^{1/2}(1) - x\left(\frac{1}{2}\right)\left(x^2+1\right)^{-1/2}(2x)}{\left[\left(x^2+1\right)^{1/2}\right]^2} = \frac{\left(x^2+1\right)^{-1/2}\left[x^2+1-x^2\right]}{x^2+1} = \frac{1}{\left(x^2+1\right)^{3/2}},$$ which is never equal to zero. We compute $f(x)$ at the endpoints, and conclude that $f(-1) = -\frac{\sqrt{2}}{2}$ is the absolute minimum value and $f(1) = \frac{\sqrt{2}}{2}$ is the absolute maximum value.

38. $g(x) = x\left(4-x^2\right)^{1/2}$ on $[0, 2]$, so

$$g'(x) = \left(4-x^2\right)^{1/2} + x\left(\tfrac{1}{2}\right)\left(4-x^2\right)^{-1/2}(-2x) = \left(4-x^2\right)^{-1/2}\left(4-x^2-x^2\right) = -\frac{2\left(x^2-2\right)}{\sqrt{4-x^2}}.$$

The critical number of g in $(0, 2)$ is $\sqrt{2}$. From the table, we conclude that $g\left(\sqrt{2}\right) = 2$ is the absolute maximum value and $g(0) = g(2) = 0$ is the absolute minimum value.

x	0	$\sqrt{2}$	2
$g(x)$	0	2	0

39. $h(t) = -16t^2 + 64t + 80$. To find the maximum value of h, we solve $h'(t) = -32t + 64 = -32(t-2) = 0$, giving $t = 2$ as the critical number of h. Furthermore, this value of t gives rise to the absolute maximum value of h since the graph of h is a parabola that opens downward. The maximum height is given by $h(2) = -16(4) + 64(2) + 80 = 144$, or 144 feet.

40. $P(x) = -10x^2 + 1760x - 50{,}000$, so $P'(x) = -20x + 1760 = 0$ if $x = 88$, and this is a critical number of P. Now $P(88) = -10(88)^2 + 1760(88) - 50{,}000 = 27{,}440$. The graph of P is a parabola that opens downward, so the point $(88, 27440)$ is an absolute maximum of P. The maximum monthly profit is \$27,440, when 88 units are rented out.

41. $f(t) = 0.136t^2 + 0.127t + 18.1$, so $f'(t) = 0.272t + 0.127$. Setting $f'(t) = 0$ gives $0.272t = -0.127$, so $t \approx -0.467$. This value of t lies outside the interval $[0, 4]$, so f has no critical number on that interval. From the table, we conclude that the lowest and highest strikeout rates are 18.1% and 20.8%, occurring in 2009 and 2013 respectively.

t	0	4
$f(t)$	18.1	20.784

42. $N(t) = -2.65t^2 + 13.13t + 39.9$, so $N'(t) = -5.3t + 13.13$. Setting $f(t) = 0$ gives $5.3t = 13.13$, so $t \approx 2.48$ is a critical number of N. From the table, we see that global iPod sales peaked at this value of t, in mid-2009, at a sales level of approximately 56.2 million units.

t	0	2.5	4
$N(t)$	39.9	56.2	50.02

43. Observe that f is continuous on $[0, 4]$. Next, we compute $f'(t) = \frac{d}{dt}\left(20t - 40t^{1/2} + 50\right) = 20 - 20t^{-1/2} = 20t^{-1/2}\left(t^{1/2} - 1\right) = 20\frac{\sqrt{t} - 1}{\sqrt{t}}$. Observe that $t = 1$ is the only critical number of f in $(0, 4)$. Because $f(0) = 50$, $f(1) = 30$, and $f(4) = 50$, we conclude that f attains its minimum value of 30 at $t = 1$. This tells us that the traffic is moving at the slowest rate at 7 a.m. and the average speed of a vehicle at that time is 30 mph.

44. The revenue is $R(x) = px = \left(\frac{100{,}000}{250 + x} - 100\right)x = \frac{100{,}000x}{250 + x} - 100x$, so $R'(x) = 100{,}000 \cdot \frac{(250 + x)(1) - x(1)}{(250 + x)^2} - 100 = \frac{25{,}000{,}000}{(250 + x)^2} - 100$. Setting $R'(x) = 0$ gives $25{,}000{,}000 = 100(250 + x)^2$, so $250 + x = \pm\sqrt{250{,}000} = \pm 500$; that is, $x = -750$ or 250. Thus, R has the critical number 250 on the interval $[0, 750]$. From the table, we see that selling 250 handbags per day gives the maximum daily profit of \$25,000.

x	0	250	750
$R(x)$	0	25,000	0

45. $h(t) = -\frac{1}{3}t^3 + 4t^2 + 20t + 2$, so $h'(t) = -t^2 + 8t + 20 = -\left(t^2 - 8t - 20\right) = -(t - 10)(t + 2) = 0$ if $t = -2$ or $t = 10$. Rejecting the negative root, we take $t = 10$. Next, we compute $h''(t) = -2t + 8$. Because $h''(10) = -20 + 8 = -12 < 0$, the Second Derivative Test indicates that the point $t = 10$ gives a relative maximum. From physical considerations, or from a sketch of the graph of h, we conclude that the rocket attains its maximum altitude at $t = 10$ with a maximum height of $h(10) = -\frac{1}{3}(10)^3 + 4(10)^2 + 20(10) + 2$, or approximately 268.7 ft.

46. $P(x) = -0.000002x^3 + 6x - 400$, so $P'(x) = -0.000006x^2 + 6 = 0$ if $x = \pm 1000$. We reject the negative root. Next, we compute $P''(x) = -0.000012x$. Because $P''(1000) = -0.012 < 0$, the Second Derivative Test shows that $x = 1000$ gives a relative maximum of f. From physical considerations, or from a sketch of the graph of f, we see that the maximum profit is realized if 1000 cases are produced per day. That profit is $P(1000) = -0.000002(1000)^3 + 6(1000) - 400$, or \$3600/day.

47. The revenue is $R(x) = px = -0.00042x^2 + 6x$. Therefore, the profit is $P(x) = R(x) - C(x) = -0.00042x^2 + 6x - \left(600 + 2x - 0.00002x^2\right) = -0.0004x^2 + 4x - 600$. $P'(x) = -0.0008x + 4 = 0$ if $x = 5000$, a critical number of P. From the table, we see that Phonola should produce 5000 discs/month.

x	0	5000	12,000
$P(x)$	-600	9400	$-10{,}200$

48. The revenue is $R(x) = px = -0.0004x^2 + 10x$ and the profit is $P(x) = R(x) - C(x) = -0.0004x^2 + 10x - \left(400 + 4x + 0.0001x^2\right) = -0.0005x^2 + 6x - 400$. $P'(x) = -0.001x + 6 = 0$ if $x = 6000$, a critical number. Because $P''(x) = -0.001 < 0$ for all x, we see that the graph of P is a parabola that opens downward. Therefore, a level of production of 6000 rackets/day will yield a maximum profit.

49. The cost function is $C(x) = V(x) + 20{,}000 = 0.000001x^3 - 0.01x^2 + 50x + 20{,}000$, so the profit function is

$$P(x) = R(x) - C(x) = -0.02x^2 + 150x - 0.000001x^3 + 0.01x^2 - 50x + 20{,}000$$
$$= -0.000001x^3 - 0.01x^2 + 100x - 20{,}000.$$

We want to maximize P on $[0, 7000]$. $P'(x) = -0.000003x^2 - 0.02x + 100$. Setting $P'(x) = 0$ gives $3x^2 + 20{,}000x - 100{,}000{,}000 = 0$, so or $x = \dfrac{-20{,}000 \pm \sqrt{20{,}000^2 + 1{,}200{,}000{,}000}}{6} = -10{,}000$ or $3{,}333.33$.

Thus, $x = 3333.33$ is a critical number in the interval $[0, 7500]$. From the table, we see that a level of production of 3,333 pagers per week will yield a maximum profit of \$165,185.20 per week.

x	0	3333.33	7500
$P(x)$	−20,000	165,185.2	−254,375

50. $R(x) = px = -0.05x^2 + 600x$, so

$$P(x) = R(x) - C(x) = -0.05x^2 + 600x - \left(0.000002x^3 - 0.03x^2 + 400x + 80{,}000\right)$$
$$= -0.000002x^3 - 0.02x^2 + 200x - 80{,}000.$$

We want to maximize P on $[0, 12000]$. $P'(x) = -0.000006x^2 - 0.04x + 200$, so setting $P'(x) = 0$ gives $3x^2 + 20{,}000x - 100{,}000{,}000 = 0$ or $x = \dfrac{-20{,}000 \pm \sqrt{20{,}000^2 + 1{,}200{,}000{,}000}}{6} = -10{,}000$ or 3333.3. Thus, $x = 3333.3$ is a critical number in the interval $[0, 12000]$. From the table, we see that a level of production of 3333 units will yield a maximum profit.

x	0	3333	12,000
$P(x)$	−80,000	290,370	−4,016,000

51. The cost function is $C(x) = 0.2\left(0.01x^2 + 120\right)$ and the average cost function is $\overline{C}(x) = \dfrac{C(x)}{x} = 0.2\left(0.01x + \dfrac{120}{x}\right) = 0.002x + \frac{24}{x}$. To find the minimum average cost, we first compute $\overline{C}'(x) = 0.002 - \dfrac{24}{x^2}$. Setting $\overline{C}'(x) = 0$ gives $0.002 - \dfrac{24}{x^2} = 0$, so $x^2 = \dfrac{24}{0.002} = 12{,}000$, and thus $x \approx \pm 110$. We reject the negative root, leaving $x = 110$ as the only critical number of $\overline{C}(x)$. Because $\overline{C}''(x) = 48x^{-3} > 0$ for all $x > 0$, we see that $\overline{C}(x)$ is concave upward on $(0, \infty)$. We conclude that $\overline{C}(110) \approx 0.44$ is the absolute minimum value of $\overline{C}(x)$ and that the average cost is minimized when $x = 110$ units.

52. **a.** $\overline{C}(x) = \dfrac{C(x)}{x} = 0.0025x + 80 + \dfrac{10{,}000}{x}$.

b. $\overline{C}'(x) = 0.0025 - \dfrac{10{,}000}{x^2} = 0$ if $0.0025x^2 = 10{,}000$, or $x = 2000$. Because $\overline{C}''(x) = \dfrac{20{,}000}{x^3}$, we see that $\overline{C}''(x) > 0$ for $x > 0$ and so $\overline{C}$ is concave upward on $(0, \infty)$. Therefore, $x = 2000$ yields a minimum.

c. We solve $\overline{C}(x) = C'(x)$: $0.0025x + 80 + \dfrac{10{,}000}{x} = 0.005x + 80$, so $0.0025x^2 = 10{,}000$ and $x = 2000$.

d. It appears that we can solve the problem in two ways.

53. a. $C(x) = 0.000002x^3 + 5x + 400$, so $\overline{C}(x) = \dfrac{C(x)}{x} = 0.000002x^2 + 5 + \dfrac{400}{x}$.

b. $\overline{C}'(x) = 0.000004x - \dfrac{400}{x^2} = \dfrac{0.000004x^3 - 400}{x^2} = \dfrac{0.000004\left(x^3 - 100{,}000{,}000\right)}{x^2}$. Setting $\overline{C}'(x) = 0$ gives $x = 464$, the only critical number of $\overline{C}$. Next, $\overline{C}''(x) = 0.000004 + \dfrac{800}{x^3}$, so $\overline{C}''(464) > 0$ and by the Second Derivative Test, the point $x = 464$ gives rise to a relative minimum. Because $\overline{C}''(x) > 0$ for all $x > 0$, $\overline{C}$ is concave upward on $(0, \infty)$ and $x = 464$ gives rise to an absolute minimum of $\overline{C}$. Thus, the smallest average product cost occurs when the level of production is 464 cases per day.

c. We want to solve the equation $\overline{C}(x) = C'(x)$, that is, $0.000002x^2 + 5 + \dfrac{400}{x} = 0.000006x^2 + 5$, so $0.000004x^3 = 400$, $x^3 = 100{,}000{,}000$, and $x = 464$.

d. The results are as expected.

54. $\overline{C}(x) = \dfrac{C(x)}{x}$, so $\overline{C}'(x) = \dfrac{xC'(x) - C(x)}{x^2} = 0$. This implies that $xC'(x) - C(x) = x^2$, so $\dfrac{C(x)}{x} = C'(x)$. This shows that at a level of production where the average cost is minimized, the average cost $\dfrac{C(x)}{x}$ is equal to the marginal cost $C'(x)$.

55. a. $\overline{C}(x) = \dfrac{C(x)}{x} = 0.0025x + 80 + \dfrac{10{,}000}{x}$.

b. Using the result of Exercise 54, we set $\dfrac{C(x)}{x} = \overline{C}(x) = C'(x)$, obtaining $0.0025x + 80 + \dfrac{10{,}000}{x} = 0.005x + 80$. This is the same equation obtained in Exercise 52(b). The lowest average production cost occurs when the production level is 2000 cases per day.

c. The average cost is equal to the marginal cost when the production level is 2000 cases per day.

d. They are the same, as expected.

56. The demand equation is $p = \sqrt{800 - x} = (800 - x)^{1/2}$, so the revenue function is $R(x) = xp = x(800 - x)^{1/2}$. To find the maximum of R, we compute

$$R'(x) = \tfrac{1}{2}(800 - x)^{-1/2}(-1)(x) + (800 - x)^{1/2} = \tfrac{1}{2}(800 - x)^{-1/2}[-x + 2(800 - x)]$$
$$= \tfrac{1}{2}(800 - x)^{-1/2}(1600 - 3x).$$

Next, $R'(x) = 0$ implies $x = 800$ or $x = \frac{1600}{3}$, the critical numbers of R. From the table, we conclude that $R\left(\frac{1600}{3}\right) = 8709$ is the absolute maximum value. Therefore, the revenue is maximized by producing $\frac{1600}{3} \approx 533$ dresses.

x	0	800	$\frac{1600}{3}$
$R(x)$	0	0	8709

57. The revenue function is $R(x) = xp = \dfrac{50x}{0.01x^2 + 1}$. To find the maximum value of R, we compute $R'(x) = \dfrac{(0.01x^2 + 1)\,50 - 50x\,(0.02x)}{(0.01x^2 + 1)^2} = -\dfrac{0.5\,(x^2 - 100)}{(0.01x^2 + 1)^2}$. Now $R'(x) = 0$ implies $x = -10$, or $x = 10$. The first root is rejected since x must be greater than or equal to zero. Thus, $x = 10$ is the only critical number.

From the table, we conclude that $R(10) = 250$ is the absolute maximum value of R. Thus, the revenue is maximized by selling 10,000 watches.

x	0	10	20
$R(x)$	0	250	200

58. $A(t) = 136\left[1 + 0.25\,(t - 4.5)^2\right]^{-1} + 28$, so

$$A'(t) = 136(-1)\left[1 + 0.25\,(t - 4.5)^2\right]^{-2}(0.25)\,2\,(t - 4.5) = -\frac{68\,(t - 4.5)}{\left[1 + 0.25\,(t - 4.5)^2\right]^2}.$$

Setting $A'(t) = 0$ gives $t = 4.5$ as a critical number of A. We see that the maximum of A occurs when $t = 4.5$, that is, at 11:30 a.m.

t	0	4.5	11
$A(t)$	50.4	164	39.8

59. $f(t) = 100\left(\dfrac{t^2 - 4t + 4}{t^2 + 4}\right)$.

a. $$f'(t) = 100\left[\frac{(t^2 + 4)(2t - 4) - (t^2 - 4t + 4)(2t)}{(t^2 + 4)^2}\right] = \frac{400\,(t^2 - 4)}{(t^2 + 4)^2} = \frac{400\,(t - 2)(t + 2)}{(t^2 + 4)^2}.$$

From the sign diagram for f', we see that $t = 2$ gives a relative minimum, and we conclude that the oxygen content is the lowest 2 days after the organic waste has been dumped into the pond.

Sign of f' on t: $- - -$ on $(0, 2)$, 0 at $t = 2$, $+ + +$ for $t > 2$.

b. $$f''(t) = 400\left[\frac{(t^2 + 4)^2(2t) - (t^2 - 4)\,2\,(t^2 + 4)(2t)}{(t + 4)^4}\right] = 400\left[\frac{(2t)(t^2 + 4)(t^2 + 4 - 2t^2 + 8)}{(t^2 + 4)^4}\right]$$
$$= -\frac{800t\,(t^2 - 12)}{(t^2 + 4)^3}.$$

$f''(t) = 0$ when $t = 0$ and $t = \pm 2\sqrt{3}$. We reject $t = 0$ and $t = -2\sqrt{3}$. From the sign diagram for f'', we see that $t = 2\sqrt{3}$ gives an inflection point of f and we conclude that this is an absolute maximum. Therefore, the rate of oxygen regeneration is greatest 3.5 days after the organic waste has been dumped into the pond.

Sign of f'' on t: 0 at $t = 0$, $+ + +$ on $(0, 2\sqrt{3})$, 0 at $t = 2\sqrt{3}$, $- - -$ for $t > 2\sqrt{3}$.

60. $v'(r) = -2kr = 0$ if $r = 0$, so there is no critical number in $(0, R)$. Calculating $v(r)$ at the endpoints, we see that v has an absolute maximum of $v(0) = kR^2$, so the velocity is greatest along the central axis.

61. We compute $\overline{R}'(x) = \dfrac{xR'(x) - R(x)}{x^2}$. Setting $\overline{R}'(x) = 0$ gives $xR'(x) - R(x) = 0$, or $R'(x) = \dfrac{R(x)}{x} = \overline{R}(x)$, so a critical number of $\overline{R}$ occurs when $\overline{R}(x) = R'(x)$. Next, we compute $\overline{R}''(x) = \dfrac{x^2\left[R'(x) + xR''(x) - R'(x)\right] - \left[xR'(x) - R(x)\right](2x)}{x^4} = \dfrac{R''(x)}{x} < 0$. Thus, by the Second Derivative Test, the critical number does give the maximum revenue.

62. $N(t) = -0.1t^3 + 1.5t^2 + 100$ and $N'(t) = -0.3t^2 + 3t$. We want to maximize the function $N'(t)$. Now $N''(t) = -0.6t + 3$, so setting $N''(t) = 0$ gives $t = 5$ as the critical number of N'. $N'''(5) = -0.6 < 0$ and $t = 5$ does give rise to a maximum for $N'(t)$, that is the growth rate was maximal in 2012, as we wished to show.

63. $G(t) = -0.2t^3 + 2.4t^2 + 60$, so the growth rate is $G'(t) = -0.6t^2 + 4.8t$. To find the maximum growth rate, we compute $G''(t) = -1.2t + 4.8$. Setting $G''(t) = 0$ gives $t = 4$ as a critical number. From the table, we see that G is maximal at $t = 4$; that is, the growth rate is greatest in 2010.

t	0	4	8
$G'(t)$	0	9.6	0

64. $D(t) = -0.038898t^3 + 0.30858t^2 - 0.31849t + 0.22$, so $D'(t) = -0.116694t^2 + 0.61716t - 0.31849$. Setting $D'(t) = 0$ gives $t = \dfrac{-0.61716 \pm \sqrt{(0.61716)^2 - 4(-0.116694)(-0.31849)}}{2(-0.116694)} \approx 0.58$ or 4.71.

From the table, we see that the largest federal budget deficit over the period under consideration was approximately \$1.5 trillion in 2010.

t	0	0.58	4.71	6
$D(t)$	0.22	0.13	1.50	1.02

65. $N(t) = -87.244444t^3 - 2482.35t^2 + 46009.26t + 579185$, so $N'(t) = -261.733328t^2 - 4964.7t + 46009.26$. Setting $N'(t) = 0$ and using the quadratic formula gives
$$t = \frac{-(-4964.7) \pm \sqrt{(-4964.7)^2 - 4(-261.733328)(46009.26)}}{2(-261.73332)} \approx -25.8$$
or 6.82, so 6.82 is an approximate critical number of N. From the table, we see that the number of new prison admissions did indeed peak in 2006 ($t = 6$) at approximately 749,833.

t	2	6.82	11
$N(t)$	660,576	749,833	668,800

66. $f(t) = -0.0004401t^3 + 0.007t^2 + 0.112t + 0.28$, so $f'(t) = -0.0013203t^2 + 0.014t + 0.112$. Setting $f'(t) = 0$ and using the quadratic formula gives $t = \dfrac{-0.014 \pm \sqrt{(0.014)^2 - 4(-0.0013203)(0.112)}}{2(-0.0013203)} \approx -5.325$ or 15.93, so the only critical number of f in the relevant interval is approximately 15.93. From the table, we see that the highest rate of death from AIDS worldwide over the period from 1990 through 2011 was approximately 2.06 million per year in 2006.

t	0	15.93	21
$f(t)$	0.28	2.06	1.64

67. $S'(t) = \frac{d}{dt}\left(0.000989t^3 - 0.0486t^2 + 0.7116t + 1.46\right) = 0.002967t^2 - 0.0972t + 0.7116$. Using the quadratic formula to solve the equation $f'(t) = 0$ gives $t = \dfrac{0.0972 \pm \sqrt{(-0.0972)^2 - 4(0.002967)(0.7116)}}{2(0.002967)} \approx 11.0$ or 21.7. From the table, we see that S has an absolute maximum when $t \approx 11$. Thus, children with superior intelligence have a cortex that reaches maximum thickness around 11 years of age.

t	5	11.0	19
$S(t)$	3.9	4.7	4.2

68. $A'(t) = \dfrac{d}{dt}\left(-0.00005t^3 - 0.000826t^2 + 0.0153t + 4.55\right) = -0.00015t^2 - 0.001652t + 0.0153$. Using the quadratic formula to solve $f'(t) = 0$ with $a = -0.00015$, $b = -0.001652$, and $c = 0.0153$, we have

$$t = \frac{-(-0.001652) \pm \sqrt{(-0.001652)^2 - 4(-0.00015)(0.0153)}}{2(-0.00015)} \approx -17.01 \text{ or } 5.997.$$

From the table, we see that A has an absolute maximum when $t \approx 6$, so the cortex of children of average intelligence reaches a maximum thickness around the time the children are 6 years old.

t	5	6	19
$A(t)$	4.5996	4.601	4.200

69. a. $P(t) = 0.00074t^3 - 0.0704t^2 + 0.89t + 6.04$, so $P'(t) = 0.00222t^2 - 0.1408t + 0.89 = 0$ implies $t = \dfrac{0.1408 \pm \sqrt{(0.1408)^2 - 4(0.00222)(0.89)}}{2(0.00222)} \approx 7.12$ or 56.3. The root 56.3 is rejected because it lies outside the interval [0, 10]. $P''(t) = 0.00444t - 0.1408$ and $P''(7.12) = -0.109 < 0$, and so $t \approx 7.12$ gives a relative maximum. This occurs around 2071.

b. The population will peak at $P(7.12) \approx 9.075$ billion.

70. a. On [0, 3], $f(t) = 0.6t^2 + 2.4t + 7.6$, so $f'(t) = 1.2t + 2.4 = 0$ implies $t = -2$ which lies outside the interval [0, 3]. (We evaluate f at each relevant point below.)

On [3, 5], $f(t) = 3t^2 + 18.8t - 63.2$, so $f'(t) = 6t + 18.8 = 0$ implies $t = -3.13$ which lies outside the interval [3, 5].

On [5, 8], $f(t) = -3.3167t^3 + 80.1t^2 - 642.583t + 1730.8025$, so $f'(t) = -9.9501t^2 + 160.2t - 642.583 = 0$ implies $t = \dfrac{-160.2 \pm \sqrt{160.2^2 - 4(-9.9501)(642.583)}}{2(-9.9501)} \approx 7.58$ or 8.52. Only the critical number $t = 7.58$ lies inside the interval [5, 8].

From the table, we see that the investment peaked when $t = 5$, that is, in the year 2000. The amount invested was \$105.8 billion.

t	0	3	5	7.58	8
$f(t)$	7.6	20.2	105.8	17.8	18.4

b. Investment was lowest (at \$7.6 billion) when $t = 0$.

71. We want to minimize the function $E(v) = \dfrac{aLv^3}{v-u}$. Because $v > u$, the function has no points of discontinuity. To find the critical numbers of $E(v)$, we solve the equation $E'(v) = \dfrac{(v-u)\,3aLv^2 - aLv^3}{(v-u)^2} = \dfrac{aLv^2(2v-3u)}{(v-u)^2} = 0$, obtaining $v = \frac{3}{2}u$ or $v = 0$. Now $v \neq 0$ since $u < v$, so $v = \frac{3}{2}u$ is the only critical number of interest. Because $E'(v) < 0$ if $v < \frac{3}{2}u$ and $E'(v) > 0$ if $v > \frac{3}{2}u$, we see that $v = \frac{3}{2}u$ gives a relative minimum. The nature of the problem suggests that $v = \frac{3}{2}u$ gives the absolute minimum of E (we can verify this by sketching the graph of E). Therefore, the fish must swim at $\frac{3}{2}u$ ft/sec in order to minimize the total energy expended.

72. $R = D^2\left(\dfrac{k}{2} - \dfrac{D}{3}\right) = \dfrac{kD^2}{2} - \dfrac{D^3}{3}$, so $\dfrac{dR}{dD} = \dfrac{2kD}{2} - \dfrac{3D^2}{3} = kD - D^2 = D(k-D)$. Setting $\dfrac{dR}{dD} = 0$, we have $D = 0$ or $k = D$. We consider only $k = D$ because $D > 0$. If $k > 0$, $\dfrac{dR}{dD} > 0$ and if $k < 0$, $\dfrac{dR}{dD} < 0$. Therefore $k = D$ gives a relative maximum. The nature of the problem suggests that $k = D$ gives the absolute maximum of R. We can also verify this by graphing R.

73. $\dfrac{dR}{dD} = kD - D^2$ and $\dfrac{d^2R}{dD^2} = k - 2D$. Setting $\dfrac{d^2R}{dD^2} = 0$, we obtain $k = 2D$, or $D = \dfrac{k}{2}$. Because $\dfrac{d^2R}{dD^2} > 0$ for $k < 2D$ and $\dfrac{d^2R}{dD^2} < 0$ for $k > 2D$, we see that $k = 2D$ provides the relative (and absolute) maximum.

74. $R'(x) = \dfrac{d}{dx}[kx(Q-x)] = k\dfrac{d}{dx}(Qx - x^2) = k(Q-2x)$ is continuous everywhere and has a zero at $\frac{1}{2}Q$; this is the only critical number of R in $(0, Q)$. $R(0) = 0$, $R\left(\frac{1}{2}Q\right) = \frac{1}{4}kQ^2$, and $R(Q) = 0$, so the absolute maximum value of R is $R\left(\frac{1}{2}Q\right) = \frac{1}{4}kQ^2$, showing that the rate of chemical reaction is greatest when exactly half of the original substrate has been transformed.

75. Setting $P' = 0$ gives $P' = \dfrac{d}{dR}\left[\dfrac{E^2R}{(R+r)^2}\right] = E^2\left[\dfrac{(R+r)^2 - R(2)(R+r)}{(R+r)^4}\right] = \dfrac{E^2(r-R)}{(R+r)^3} = 0$. Therefore, $R = r$ is a critical number of P. Because $P'' = E^2\dfrac{(R+r)^3(-1) - (r-R)(3)(R+r)^2}{(R+r)^6} = \dfrac{2E^2(R-2r)}{(R+r)^4}$ and $P''(r) = \dfrac{-2E^2r}{(2r)^4} = -\dfrac{E^2}{8r^3} < 0$, the Second Derivative Test and physical considerations both imply that $R = r$ gives a relative maximum value of P. The maximum power is $P = \dfrac{E^2r}{(2r)^2} = \dfrac{E^2}{4r}$ watts.

76. Setting $v' = 0$ gives

$$v' = \frac{d}{dL}\left[k\sqrt{\frac{L}{C} + \frac{C}{L}}\right] = k\frac{d}{dL}\left(\frac{L}{C} + \frac{C}{L}\right)^{1/2} = \frac{k}{2}\left(\frac{L}{C} + \frac{C}{L}\right)^{-1/2}\left(\frac{1}{C} - \frac{C}{L^2}\right) = \frac{k(L^2 - C^2)}{2CL^2\sqrt{\dfrac{L}{C} + \dfrac{C}{L}}} = 0.$$

Therefore, $L = \pm C$. Because v is not defined for $L = -C$, we reject that root. The length of the wave with minimum velocity is C.

77. $\frac{dx}{dt} = \frac{d}{dt}\left[1.5(10-t) - 0.0013(10-t)^4\right] = -1.5 - 0.0013(4)(10-t)^3(-1) = -1.5 + 0.0052(10-t)^3$ is continuous everywhere and has zeros where $0.0052(10-t^3) = 1.5$; that is, $(10-t)^3 = \frac{1.5}{0.0052}$, or $t = 10 - \sqrt[3]{\frac{1.5}{0.0052}} \approx 3.4$, and so x has the critical number 3.4 in $(0, 10)$. Now $x(0) = 2$, $x(3.4) = 7.4$, and $x(10) = 0$, showing that after 3.4 minutes, the maximum amount of salt (roughly 7.4 lb) is in the tank.

78. False. Let $f(x) = \begin{cases} -x & \text{if } -1 \le x < 0 \\ \frac{1}{2} & \text{if } 0 \le x < 1 \end{cases}$ Then f is discontinuous at $x = 0$, but f has an absolute maximum value of 1, attained at $x = -1$.

79. False. Let $f(x) = \begin{cases} |x| & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$ on $[-1, 1]$.

80. True. $f''(x) < 0$ on (a, b), so the graph of f is concave downward on (a, b). Therefore, the relative maximum value at $x = c$ must be the absolute maximum value.

81. True. The absolute extrema of f must occur for some x in (a, b) at which $f'(x) = 0$, or at an endpoint. Since $f'(x) \ne 0$ for all x in (a, b), the absolute extrema of f (and in particular its absolute maximum) must occur at $x = a$ or $x = b$, with value $f(a)$ or $f(b)$.

82. True. Since $f'(x) > 0$ for all x in (a, b), we see that the absolute extrema of f must occur at $x = a$ or $x = b$ (see Exercise 81). But f is increasing on (a, b), which implies that the absolute minimum value of f must occur at the left endpoint a.

83. True. This follows from the Second Derivative Test applied to the function $P = R - C$.

84. False. Consider $f(x) = 1/x$ on the interval $(0, \infty)$.

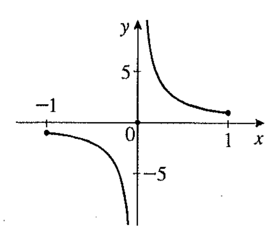
85. Because $f(x) = c$ for all x, the function f satisfies $f(x) \le c$ for all x and so f has absolute maxima at all values of x. Similarly, f has absolute minima at all values of x.

86. Suppose f is a nonconstant polynomial function. Then $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \ne 0$ and $n \ge 1$.
First, let us suppose that $a_n > 0$. There are two cases to consider:
(1) If n is odd, then $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, and so f has no absolute extremum.
(2) If n is even, then $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$, so f cannot have an absolute maximum.
A similar argument is used in the case where $a_n < 0$.

87. **a.** f is not continuous at $x = 0$ because $\lim_{x\to 0} f(x)$ does not exist.

b. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty$.

c.



88. $f(x)$ can be made as close to -1 as we please by taking x sufficiently close to -1. But the value -1 is never attained, because x must be greater than -1. Similarly, 1 is never attained. Therefore, f has neither an absolute minimum value nor an absolute maximum value.

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1. Absolute maximum value 145.9, absolute minimum value -4.3834.

2. Absolute maximum value 26.3997, absolute minimum value -4.4372.

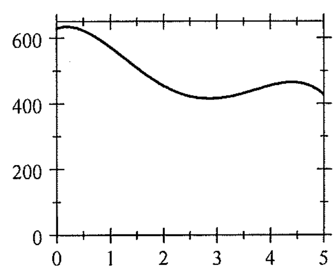
3. Absolute maximum value 16, absolute minimum value -0.1257.

4. Absolute maximum value 11.2016, absolute minimum value 9.

5. Absolute maximum value 2.8889, absolute minimum value 0.

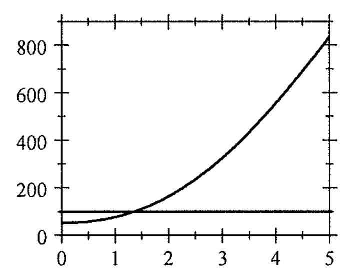
6. No absolute maximum or minimum value.

7. a.



b. Using the function for finding the absolute minimum of f on $[0, 5]$, we see that the absolute minimum value of f is approximately 415.56, occurring when $x \approx 2.87$. This proves the assertion.

8. a. The graphs of $y_1 = g(t)$ and $y_2 = 100$ are shown below.



The graphs intersect at approximately $(1.36, 100)$. This says that the construction loans of peer banks first exceeded the recommended maximum of 100% near the beginning of May 2004.

b. The maximum was approximately $g(5) \approx 836\%$.