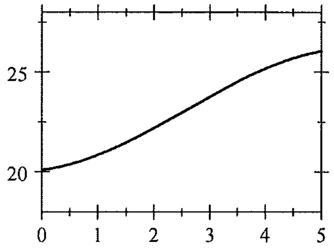
12. a. $y = -0.06204t^3 + 0.48135t^2 + 0.3354t + 20.1, 0 \le t \le 5.$

b.



c. From the graph, it appears that f is increasing on $(0, 5)$. This tells us that the age at first marriage is increasing from 1960 through 2010.

d. $y' = -0.18612t^2 + 0.9627t + 0.3354$. Setting $f'(t) = 0$ and solving for t gives $t \approx -0.328$ or 5.50. Both numbers lie outside the domain of f, and because $f'(0) = 0.3354 > 0$, we conclude that $f'(t) > 0$ on $(0, 5)$. Thus, f is increasing on $(0, 5)$.

4.2 Applications of the Second Derivative

Concept Questions page 282

1. a. f is concave upward on (a, b) if f' is increasing on (a, b). f is concave downward on (a, b) if f' is decreasing on (a, b).

b. For the procedure for determining where f is concave upward and where f is concave downward, see page 274 of the text.

2. An inflection point of the graph of f is a point on the graph of f where its concavity changes from upward to downward or vice versa. See page 276 of the text for a procedure for finding inflection points.

3. The Second Derivative Test is stated on page 280 of the text. In general, if f'' is easy to compute, then use the Second Derivative Test. However, keep in mind that (1) in order to use this test f'' must exist, (2) the test is inconclusive if $f''(c) = 0$, and (3) the test is inconvenient to use if f'' is difficult to compute.

Exercises page 282

1. f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$. f has an inflection point at $(0, 0)$.

2. f is concave downward on $\left(0, \frac{3}{2}\right)$ and concave upward on $\left(\frac{3}{2}, \infty\right)$. f has an inflection point at $\left(\frac{3}{2}, 2\right)$.

3. f is concave downward on $(-\infty, 0)$ and $(0, \infty)$.

4. f is concave upward on $(-\infty, -4)$ and $(4, \infty)$ and concave downward on $(-4, 4)$.

5. f is concave upward on $(-\infty, 0)$ and $(1, \infty)$ and concave downward on $(0, 1)$. $(0, 0)$ and $(1, -1)$ are inflection points of f.

6. f is concave upward on $(0, 1)$ and $(5, \infty)$ and concave downward on $(1, 5)$.

7. f is concave downward on $(-\infty, -2)$ and $(-2, 2)$ and $(2, \infty)$.

8. f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$. $(0, 1)$ is an inflection point.

9. a. f is concave upward on $(0, 2)$, $(4, 6)$, $(7, 9)$, and $(9, 12)$ and concave downward on $(2, 4)$ and $(6, 7)$.

b. f has inflection points at $\left(2, \frac{5}{2}\right)$, $(4, 2)$, $(6, 2)$, and $(7, 3)$.

10. **a.** $f'(3) = 0$ and so 3 is a critical number of f. Also, $f''(3) < 0$, and so the Second Derivative Test applies and tells us that 3 gives a relative maximum. $f'(5) = 0$ and so 5 is a critical number of f. Also, $f''(5) > 0$, and so 5 gives a relative minimum according to the SDT.

b. The Second Derivative Test is not conclusive when applied to the critical number 7 because $f''(7) = 0$. It cannot be used at the critical number 9 because $f''(9)$ does not exist.

11. (a) **12.** (b) **13.** (b) **14.** (c)

15. **a.** $D_1'(t) > 0$, $D_2'(t) > 0$, $D_1''(t) > 0$, and $D_2''(t) < 0$ on $(0, 12)$.

b. With or without the proposed promotional campaign, the deposits will increase, but with the promotion, the deposits will increase at an increasing rate whereas without the promotion, the deposits will increase at a decreasing rate.

16. **a.** The graph of f is concave upward on $(0, t_1)$, indicating that Car A is accelerating for $0 < t < t_1$. The graph of g is concave downward on $(0, t_1)$, indicating that Car B is decelerating for $0 < t < t_1$.

b. The graph of f is concave downward on (t_1, t_2), indicating that Car A is decelerating for $t_1 < t < t_2$. The graph of g is concave upward on (t_1, t_2), indicating that Car B is accelerating for $t_1 < t < t_2$.

c. f has an inflection point at t_1, so the acceleration of Car A is greatest at $t = t_1$. g has an inflection point at t_1, so the acceleration of Car B is least at t_1.

d. The two cars have the same velocities at $t = 0$, $t = t_1$, and $t = t_2$.

17. (c) **18.** (a) **19.** (d) **20.** (b)

21. **a.** Between 8 a.m. and 10 a.m. the rate of change of the rate of smartphone assembly is increasing; between 10 a.m. and 12 noon, that rate is decreasing.

b. If you look at the tangent lines to the graph of N, you will see that the tangent line at P has the greatest slope. This means that the rate at which the average worker is assembling smartphones is greatest—that is, the worker is most efficient—when $t = 2$, at 10 a.m.

22. **a.** $f'(t)$ is negative on $(0, t_2)$ and positive on (t_2, t_3). $f''(t)$ is negative on $(0, t_1)$ and positive on (t_1, t_3).

b. The graph of f has an inflection point at $(t_1, f(t_1))$.

c. Total deposits were initially decreasing faster and faster. At time t_1, the rate of decrease reached its peak and began to lessen. At time t_2, deposits reached their lowest total and began to rebound.

23. The significance of the inflection point Q is that the restoration process is working at its peak at the time t_0 corresponding to the t-coordinate of Q.

24. **a.** The total change in the index over the period in question is zero.

b. The index was highest in June 2005.

c. The index was increasing at the greatest rate in June 2003 and decreasing at the greatest rate in June 2006.

25. $f(x) = 4x^2 - 12x + 7$, so $f'(x) = 8x - 12$ and $f''(x) = 8$. Thus, $f''(x) > 0$ everywhere, and so f is concave upward everywhere.

26. $g(x) = x^4 + \frac{1}{2}x^2 + 6x + 10$, so $g'(x) = 4x^3 + x + 6$ and $g''(x) = 12x^2 + 1$. We see that $g''(x) \geq 1$ for all values of x, and so g is concave upward everywhere.

27. $f(x) = \dfrac{1}{x^4} = x^{-4}$, so $f'(x) = -\dfrac{4}{x^5}$ and $f''(x) = \dfrac{20}{x^6} > 0$ for all values of x in $(-\infty, 0)$ and $(0, \infty)$, and so f is concave upward on its domain.

28. $g(x) = -\sqrt{4-x^2}$, so $g'(x) = \dfrac{d}{dx}\left[-\left(4-x^2\right)^{1/2}\right] = -\frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x) = x\left(4-x^2\right)^{-1/2}$ and

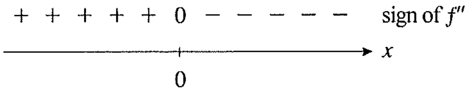
$$g''(x) = \left(4-x^2\right)^{-1/2} + x\left(-\tfrac{1}{2}\right)\left(4-x^2\right)^{-3/2}(-2x) = \left(4-x^2\right)^{-3/2}\left[\left(4-x^2\right)+x^2\right] = \frac{4}{\left(4-x^2\right)^{3/2}} > 0$$

whenever it is defined. Thus, g is concave upward wherever it is defined.

29. $f(x) = 2x^2 - 3x + 4$, so $f'(x) = 4x - 3$ and $f''(x) = 4x > 0$ for all values of x. Thus, f is concave upward on $(-\infty, \infty)$.

30. $g(x) = -x^2 + 3x + 4$, so $g'(x) = -2x + 3$ and $g''(x) = -2 < 0$ for all values of x. Thus, g is concave downward on $(-\infty, \infty)$.

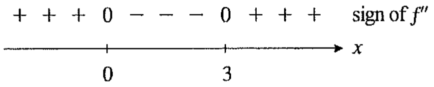
31. $f(x) = 1 - x^3$, so $f'(x) = -3x^2$ and $f''(x) = -6x$. From the sign diagram of f'', we see that f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.



32. $g(x) = x^3 - x$, so $g'(x) = 3x^2 - 1$ and $g''(x) = 6x$. Because $g''(x) < 0$ if $x < 0$ and $g''(x) > 0$ if $x > 0$, we see that g is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.

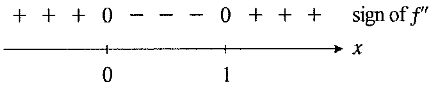
33. $f(x) = x^4 - 6x^3 + 2x + 8$, so $f'(x) = 4x^3 - 18x^2 + 2$ and $f''(x) = 12x^2 - 36x = 12x(x-3)$.

The sign diagram of f'' shows that f is concave upward on $(-\infty, 0)$ and $(3, \infty)$ and concave downward on $(0, 3)$.



34. $f(x) = 3x^4 - 6x^3 + x - 8$, so $f'(x) = 12x^3 - 18x^2 + 1$ and $f''(x) = 36x^2 - 36x = 36x(x-1)$.

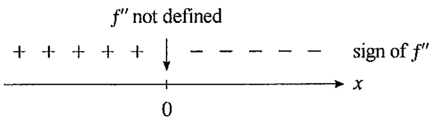
From the sign diagram of f'', we conclude that f is concave upward on $(-\infty, 0)$ and $(1, \infty)$ and concave downward on $(0, 1)$.



35. $f(x) = x^{4/7}$, so $f'(x) = \frac{4}{7}x^{-3/7}$ and $f''(x) = -\frac{12}{49}x^{-10/7} = -\dfrac{12}{49x^{10/7}}$. Observe that $f''(x) < 0$ for all $x \neq 0$, so f is concave downward on $(-\infty, 0)$ and $(0, \infty)$.

36. $f(x) = x^{1/3}$, so $f'(x) = \frac{1}{3}x^{-2/3}$ and

$f''(x) = -\frac{2}{9}x^{-5/3} = -\dfrac{2}{9x^{5/3}}$. From the sign diagram of f'', we see that f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$.

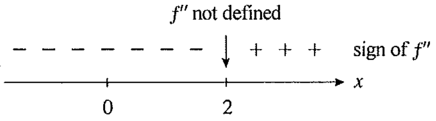


37. $f(x) = (4-x)^{1/2}$, so $f'(x) = \frac{1}{2}(4-x)^{-1/2}(-1) = -\frac{1}{2}(4-x)^{-1/2}$ and

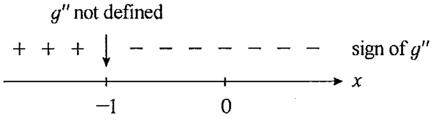
$f''(x) = \frac{1}{4}(4-x)^{-3/2}(-1) = -\dfrac{1}{4(4-x)^{3/2}} < 0$ whenever it is defined, so f is concave downward on $(-\infty, 4)$.

38. $g(x)=\sqrt{x-2}=(x-2)^{1/2}$, so $g'(x)=\frac{1}{2}(x-2)^{-1/2}$ and $g''(x)=-\frac{1}{4}(x-2)^{-3/2}=-\frac{1}{4(x-2)^{3/2}}$, which is negative for $x>2$. Next, the domain of g is $[2,\infty)$, and we conclude that g is concave downward on $(2,\infty)$.

39. $f'(x)=\frac{d}{dx}(x-2)^{-1}=-(x-2)^{-2}$ and $f''(x)=2(x-2)^{-3}=\frac{2}{(x-2)^3}$. The sign diagram of f''shows that f is concave downward on $(-\infty,2)$ and concave upward on $(2,\infty)$.



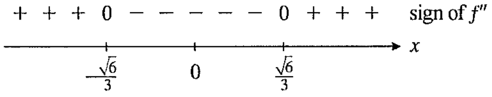
40. $g(x)=\frac{x}{x+1}$, so $g'(x)=\frac{(x+1)(1)-x(1)}{(x+1)^2}=\frac{1}{(x+1)^2}=(x+1)^{-2}$ and $g''(x)=-2(x+1)^{-3}=-\frac{2}{(x+1)^3}$. The sign diagram of $g''(x)$ shows that g is concave upward on $(-\infty,-1)$ and concave downward on $(-1,\infty)$.



41. $f'(x)=\frac{d}{dx}(2+x^2)^{-1}=-(2+x^2)^{-2}(2x)=-2x(2+x^2)^{-2}$ and

$$f''(x)=-2(2+x^2)^{-2}-2x(-2)(2+x^2)^{-3}(2x)=2(2+x^2)^{-3}\left[-(2+x^2)+4x^2\right]=\frac{2(3x^2-2)}{(2+x^2)^3}=0 \text{ if}$$

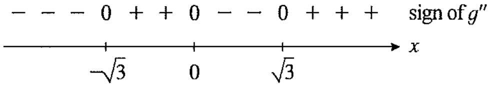
$x=\pm\sqrt{\frac{2}{3}}=\frac{\sqrt{6}}{3}$. From the sign diagram of f'', we see that f is concave upward on $\left(-\infty,-\frac{\sqrt{6}}{3}\right)$ and $\left(\frac{\sqrt{6}}{3},\infty\right)$ and concave downward on $\left(-\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{3}\right)$.



42. $g(x)=\frac{x}{1+x^2}$, so $g'(x)=\frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}$ and

$$g''(x)=\frac{(1+x^2)^2(-2x)-(1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}=\frac{-2x(1+x^2)(1+x^2+2-2x^2)}{(1+x^2)^4}=-\frac{2x(3-x^2)}{(1+x^2)^3}.$$

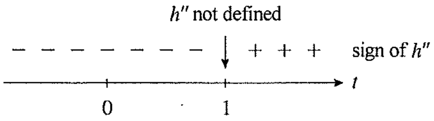
The sign diagram for g'' shows that g is concave downward on $\left(-\infty,-\sqrt{3}\right)$ and $\left(0,\sqrt{3}\right)$ and concave upward on $\left(-\sqrt{3},0\right)$ and $\left(\sqrt{3},\infty\right)$.



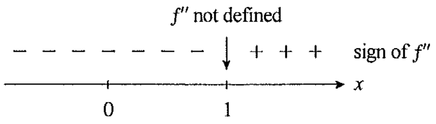
43. $h(t) = \frac{t^2}{t-1}$, so $h'(t) = \frac{(t-1)(2t) - t^2(1)}{(t-1)^2} = \frac{t^2-2t}{(t-1)^2}$ and

$$h''(t) = \frac{(t-1)^2(2t-2) - (t^2-2t)\,2(t-1)}{(t-1)^4} = \frac{(t-1)(2t^2-4t+2-2t^2+4t)}{(t-1)^4} = \frac{2}{(t-1)^3}.$$

The sign diagram of h'' shows that h is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.



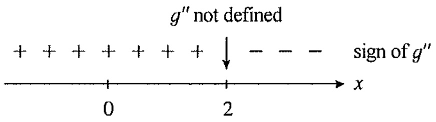
44. $f(x) = \frac{x+1}{x-1}$, so $f'(x) = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = -\frac{2}{(x-1)^2} = -2(x-1)^{-2}$ and $f''(x) = (-2)(-2)(x-1)^{-3} = \frac{4}{(x-1)^3}$. The sign diagram of f'' shows that f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.



45. $g(x) = x + \frac{1}{x^2}$, so $g'(x) = 1 - 2x^{-3}$ and $g''(x) = 6x^{-4} = \frac{6}{x^4} > 0$ whenever $x \neq 0$. Therefore, g is concave upward on $(-\infty, 0)$ and $(0, \infty)$.

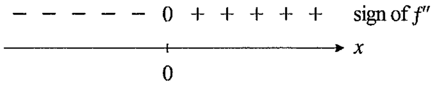
46. $h(r) = -(r-2)^{-2}$, so $h'(r) = 2(r-2)^{-3}$ and $h''(r) = -6(r-2)^{-4} < 0$ for all $r \neq 2$. Thus, h is concave downward on $(-\infty, 2)$ and $(2, \infty)$.

47. $g(t) = (2t-4)^{1/3}$, so $g'(t) = \frac{1}{3}(2t-4)^{-2/3}(2) = \frac{2}{3}(2t-4)^{-2/3}$ and $g''(t) = -\frac{4}{9}(2t-4)^{-5/3} = -\frac{4}{9(2t-4)^{5/3}}$. The sign diagram of g'' shows that g is concave upward on $(-\infty, 2)$ and concave downward on $(2, \infty)$.



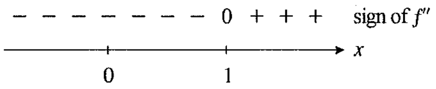
48. $f(x) = (x-2)^{2/3}$, so $f'(x) = \frac{2}{3}(x-2)^{-1/3}$ and $f''(x) = -\frac{2}{9}(x-2)^{-4/3} = -\frac{2}{9(x-2)^{4/3}} < 0$ for all $x \neq 2$. Therefore, f is concave downward on $(-\infty, 2)$ and $(2, \infty)$.

49. $f(x) = x^3 - 2$, so $f'(x) = 3x^2$ and $f''(x) = 6x$. $f''(x)$ is continuous everywhere and has a zero at $x = 0$. From the sign diagram of f'', we conclude that $(0, -2)$ is an inflection point of f.



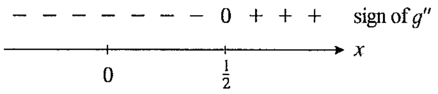
50. $g(x) = x^3 - 6x$, so $g'(x) = 3x^2 - 6$ and $g''(x) = 6x$. Observe that $g''(x) = 0$ if $x = 0$. Because $g''(x) < 0$ if $x < 0$ and $g''(x) > 0$ if $x > 0$, we see that $(0, 0)$ is an inflection point of g.

51. $f(x) = 6x^3 - 18x^2 + 12x - 20$, so $f'(x) = 18x^2 - 36x + 12$ and $f''(x) = 36x - 36 = 36(x-1) = 0$ if $x = 1$. The sign diagram of f'' shows that f has an inflection point at $(1, -20)$.

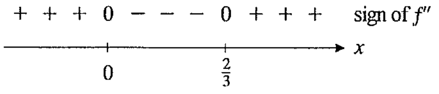


4.2

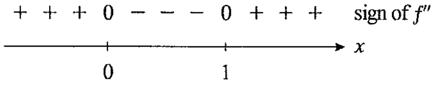
52. $g(x)=2x^3-3x^2+18x-8$, so $g'(x)=6x^2-6x+18$ and $g''(x)=12x-6=6(2x-1)$. From the sign diagram of g'', we conclude that $\left(\frac{1}{2},\frac{1}{2}\right)$ is an inflection point of g.



53. $f(x)=3x^4-4x^3+1$, so $f'(x)=12x^3-12x^2$ and $f''(x)=36x^2-24x=12x(3x-2)=0$ if $x=0$ or $\frac{2}{3}$. These are candidates for inflection points. The sign diagram of f'' shows that $(0,1)$ and $\left(\frac{2}{3},\frac{11}{27}\right)$ are inflection points of f.



54. $f(x)=x^4-2x^3+6$, so $f'(x)=4x^3-6x^2$ and $f''(x)=12x^2-12x=12x(x-1)$. $f''(x)$ is continuous everywhere and has zeros at $x=0$ and $x=1$. From the sign diagram of f'', we conclude that $(0,6)$ and $(1,5)$ are inflection points of f.



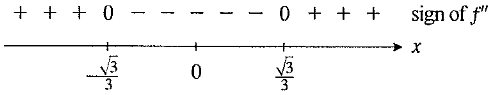
55. $g(t)=t^{1/3}$, so $g'(t)=\frac{1}{3}t^{-2/3}$and $g''(t)=-\frac{2}{9}t^{-5/3}=-\frac{2}{9t^{5/3}}$. Observe that $t=0$ is in the domain of g. Next, since $g''(t)>0$ if $t<0$ and $g''(t)<0$ if $t>0$, we see that $(0,0)$ is an inflection point of g.

56. $f(x)=x^{1/5}$, so $f'(x)=\frac{1}{5}x^{-4/5}$ and $f''(x)=-\frac{4}{25}x^{-9/5}=-\frac{4}{25x^{9/5}}$. Observe that $f''(x)>0$ if $x<0$ and $f''(x)<0$ if $x>0$. Therefore, $(0,0)$ is an inflection point.

57. $f(x)=(x-1)^3+2$, so $f'(x)=3(x-1)^2$ and $f''(x)=6(x-1)$. Observe that $f''(x)<0$ if $x<1$ and $f''(x)>0$ if $x>1$ and so $(1,2)$ is an inflection point of f.

58. $f(x)=(x-2)^{4/3}$, so $f'(x)=\frac{4}{3}(x-2)^{1/3}$ and $f''(x)=\frac{4}{9}(x-2)^{-2/3}=\frac{4}{9(x-2)^{2/3}}$. $x=2$ is a candidate for an inflection point of f, but $f''(x)>0$ for all values of x except $x=2$ and so f has no inflection point.

59. $f(x)=\frac{2}{1+x^2}=2\left(1+x^2\right)^{-1}$, so $f'(x)=-2\left(1+x^2\right)^{-2}(2x)=-4x\left(1+x^2\right)^{-2}$ and $f''(x)=-4\left(1+x^2\right)^{-2}-4x(-2)\left(1+x^2\right)^{-3}(2x)=4\left(1+x^2\right)^{-3}\left[-\left(1+x^2\right)+4x^2\right]=\frac{4\left(3x^2-1\right)}{\left(1+x^2\right)^3}$, which is continuous everywhere and has zeros at $x=\pm\frac{\sqrt{3}}{3}$. From the sign diagram of f'', we conclude that $\left(-\frac{\sqrt{3}}{3},\frac{3}{2}\right)$ and $\left(\frac{\sqrt{3}}{3},\frac{3}{2}\right)$ are inflection points of f.



60. $f(x)=2+\frac{3}{x}$, so $f'(x)=-\frac{3}{x^2}$ and $f''(x)=\frac{6}{x^3}$. Now f'' changes sign as we move across $x=0$, but $x=0$ is not in the domain of f and so f has no inflection point.

61. $f(x)=-x^2+2x+4$, so $f'(x)=-2x+2$. The critical number of f is $x=1$. Because $f''(x)=-2$ and $f''(1)=-2<0$, we conclude that $f(1)=5$ is a relative maximum of f.

62. $g(x) = 2x^2 + 3x + 7$, so $g'(x) = 4x + 3 = 0$ if $x = -\frac{3}{4}$ and this is a critical number of g. Next, $g''(x) = 4$, and so $g''\left(-\frac{3}{4}\right) = 4 > 0$. Thus, $\left(-\frac{3}{4}, \frac{47}{8}\right)$ is a relative minimum of g.

63. $f(x) = 2x^3 + 1$, so $f'(x) = 6x^2 = 0$ if $x = 0$ and this is a critical number of f. Next, $f''(x) = 12x$, and so $f''(0) = 0$. Thus, the Second Derivative Test fails. But the First Derivative Test shows that $(0, 0)$ is not a relative extremum.

64. $g(x) = x^3 - 6x$, so $g'(x) = 3x^2 - 6 = 3\left(x^2 - 2\right) = 0$ implies that $x = \pm\sqrt{2}$ are the critical numbers of g. Next, $g''(x) = 6x$. Because $g''\left(-\sqrt{2}\right) = -6\sqrt{2} < 0$ and $g''\left(\sqrt{2}\right) = 6\sqrt{2} > 0$, we conclude, by the Second Derivative Test, that $\left(-\sqrt{2}, 4\sqrt{2}\right)$ is a relative maximum and $\left(\sqrt{2}, -4\sqrt{2}\right)$ is a relative minimum of g.

65. $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x - 5$, so $f'(x) = x^2 - 4x - 5 = (x - 5)(x + 1)$ and this gives $x = -1$ and $x = 5$ as critical numbers of f. Next, $f''(x) = 2x - 4$. Because $f''(-1) = -6 < 0$, we see that $\left(-1, -\frac{7}{3}\right)$ is a relative maximum of f. Next, $f''(5) = 6 > 0$ and this shows that $\left(5, -\frac{115}{3}\right)$ is a relative minimum of f.

66. $f(x) = 2x^3 + 3x^2 - 12x - 4$, so $f'(x) = 6x^2 + 6x - 12 = 6\left(x^2 + x - 2\right) = 6(x + 2)(x - 1)$. The critical numbers of f are $x = -2$ and $x = 1$. $f''(x) = 12x + 6 = 6(2x + 1)$, so $f''(-2) = 6(-4 + 1) = -18 < 0$ and $f''(1) = 6(2 + 1) = 18 > 0$. Using the Second Derivative Test, we conclude that $f(-2) = 16$ is a relative maximum of f and $f(1) = -11$ is a relative minimum f.

67. $g(t) = t + \dfrac{9}{t}$, so $g'(t) = 1 - \dfrac{9}{t^2} = \dfrac{t^2 - 9}{t^2} = \dfrac{(t + 3)(t - 3)}{t^2}$, showing that $t = \pm 3$ are critical numbers of g. Now, $g''(t) = 18t^{-3} = \dfrac{18}{t^3}$. Because $g''(-3) = -\frac{18}{27} < 0$, the Second Derivative Test implies that g has a relative maximum at $(-3, -6)$. Also, $g''(3) = \frac{18}{27} > 0$ and so g has a relative minimum at $(3, 6)$.

68. $f(t) = 2t + 3t^{-1}$, so $f'(t) = 2 - 3t^{-2}$. Setting $f'(t) = 0$ gives $3t^{-2} = 2$ or $t^2 = \frac{3}{2}$, so $t = \pm\sqrt{\frac{3}{2}}$ are critical numbers of f. Next, we compute $f''(t) = 6/t^3$. Because $f''\left(-\sqrt{\frac{3}{2}}\right) < 0$ and $f''\left(\sqrt{\frac{3}{2}}\right) > 0$, we see that $f\left(-\sqrt{\frac{3}{2}}\right) = -2\sqrt{\frac{3}{2}} - 3\sqrt{\frac{2}{3}}$ is a relative maximum and $f\left(\sqrt{\frac{3}{2}}\right) = 2\sqrt{\frac{3}{2}} + 3\sqrt{\frac{2}{3}}$ is a relative minimum of f.

69. $f(x) = \dfrac{x}{1 - x}$, so $f'(x) = \dfrac{(1 - x)(1) - x(-1)}{(1 - x)^2} = \dfrac{1}{(1 - x)^2}$ is never zero. Thus, there is no critical number and f has no relative extremum.

70. $f(x) = \dfrac{2x}{x^2 + 1}$, so $f'(x) = \dfrac{\left(x^2 + 1\right)(2) - 2x(2x)}{\left(x^2 + 1\right)^2} = \dfrac{2\left(1 - x^2\right)}{\left(x^2 + 1\right)^2} = 0$ if $x = \pm 1$. Thus, $x = \pm 1$ are critical numbers of f. Next,

$$f''(x) = \frac{\left(x^2 + 1\right)^2(-4x) - 2\left(1 - x^2\right)2\left(x^2 + 1\right)(2x)}{\left(x^2 + 1\right)^4} = \frac{2x\left(x^2 + 1\right)\left(-2x^2 - 2 - 4 + 4x^2\right)}{\left(x^2 + 1\right)^4} = \frac{4x\left(x^2 - 3\right)}{\left(x^2 + 1\right)^3}.$$

Because $f''(-1) = \dfrac{-2(-4)}{2^3} = 1 > 0$ and $f''(1) = \dfrac{4(-2)}{2^3} = -1 < 0$, we see that f has a relative minimum at $(-1, -1)$ and a relative maximum at $(1, 1)$.

71. $f(t) = t^2 - \dfrac{16}{t}$, so $f'(t) = 2t + \dfrac{16}{t^2} = \dfrac{2t^3+16}{t^2} = \dfrac{2(t^3+8)}{t^2}$. Setting $f'(t) = 0$ gives $t = -2$ as a critical number. Next, we compute $f''(t) = \dfrac{d}{dt}\left(2t + 16t^{-2}\right) = 2 - 32t^{-3} = 2 - \dfrac{32}{t^3}$. Because $f''(-2) = 2 - \dfrac{32}{(-8)} = 6 > 0$, we see that $(-2, 12)$ is a relative minimum.

72. $g(x) = x^2 + \dfrac{2}{x}$, so $g'(x) = 2x - \dfrac{2}{x^2}$. Setting $g'(x) = 0$ gives $x^3 = 1$, or $x = 1$. Thus, $x = 1$ is the only critical number of g. Next, $g''(x) = 2 + \dfrac{4}{x^3}$. Because $g''(1) = 6 > 0$, we conclude that $g(1) = 3$ is a relative minimum of g.

73. $g(s) = \dfrac{s}{1+s^2}$, so $g'(s) = \dfrac{\left(1+s^2\right)(1) - s(2s)}{\left(1+s^2\right)^2} = \dfrac{1-s^2}{\left(1+s^2\right)^2} = 0$ gives $s = -1$ and $s = 1$ as critical numbers of g. Next, we compute

$$g''(s) = \frac{\left(1+s^2\right)^2(-2s) - \left(1-s^2\right)2\left(1+s^2\right)(2s)}{\left(1+s^2\right)^4} = \frac{2s\left(1+s^2\right)\left(-1-s^2-2+2s^2\right)}{\left(1+s^2\right)^4} = \frac{2s\left(s^2-3\right)}{\left(1+s^2\right)^3}.$$

Now $g''(-1) = \frac{1}{2} > 0$, and so $g(-1) = -\frac{1}{2}$ is a relative minimum of g. Next, $g''(1) = -\frac{1}{2} < 0$ and so $g(1) = \frac{1}{2}$ is a relative maximum of g.

74. $g'(x) = \dfrac{d}{dx}\left(1+x^2\right)^{-1} = -\left(1+x^2\right)^{-2}(2x) = -\dfrac{2x}{\left(1+x^2\right)^2}$. Setting $g'(x) = 0$ gives $x = 0$ as the only critical number. Next, we find

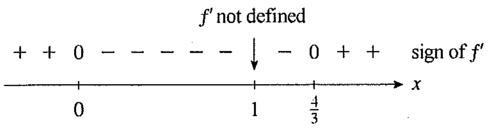
$$g''(x) = \frac{\left(1+x^2\right)^2(-2) + 2x(2)\left(1+x^2\right)(2x)}{\left(1+x^2\right)^4} = \frac{-2\left(1+x^2\right)\left(1+x^2-4x^2\right)}{\left(1+x^2\right)^4} = -\frac{2\left(1-3x^2\right)}{\left(1+x^2\right)^3}.$$

Because $g''(0) = -2 < 0$, we see that $(0, 1)$ is a relative maximum.

75. $f(x) = \dfrac{x^4}{x-1}$, so $f'(x) = \dfrac{(x-1)\left(4x^3\right) - x^4(1)}{(x-1)^2} = \dfrac{4x^4 - 4x^3 - x^4}{(x-1)^2} = \dfrac{3x^4 - 4x^3}{(x-1)^2} = \dfrac{x^3(3x-4)}{(x-1)^2}$. Thus, $x = 0$ and $x = \frac{4}{3}$ are critical numbers of f. Next,

$$\begin{aligned} f''(x) &= \frac{(x-1)^2\left(12x^3 - 12x^2\right) - \left(3x^4 - 4x^3\right)(2)(x-1)}{(x-1)^4} \\ &= \frac{(x-1)\left(12x^4 - 12x^3 - 12x^3 + 12x^2 - 6x^4 + 8x^3\right)}{(x-1)^4} \\ &= \frac{6x^4 - 16x^3 + 12x^2}{(x-1)^3} = \frac{2x^2\left(3x^2 - 8x + 6\right)}{(x-1)^3}. \end{aligned}$$

Because $f''\left(\frac{4}{3}\right) > 0$, we see that $f\left(\frac{4}{3}\right) = \frac{256}{27}$ is a relative minimum. Because $f''(0) = 0$, the Second Derivative Test fails. Using the sign diagram for f' and the First Derivative Test, we see that $f(0) = 0$ is a relative maximum.

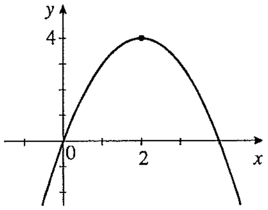


76. $f(x) = \dfrac{x^2}{x^2+1}$, so $f'(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$.

Setting $f'(x) = 0$ gives $x = 0$ as the only critical number of f.

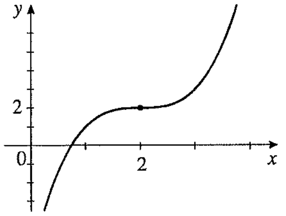
$f''(x) = \dfrac{(1+x^2)^2(2) - 2x(2)(1+x^2)(2x)}{(1+x^2)^4} = \dfrac{2(1+x^2)\left[(1+x^2) - 4x^2\right]}{(1+x^2)^4} = \dfrac{2(1-3x^2)}{(1+x^2)^3}$, so since $f''(0) = 2 > 0$, we see that $(0, 0)$ is a relative minimum.

77.



(Graph is not unique.)

78.



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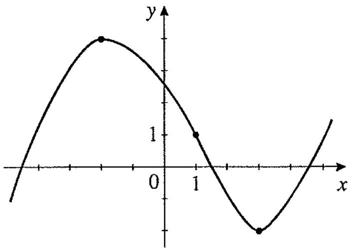
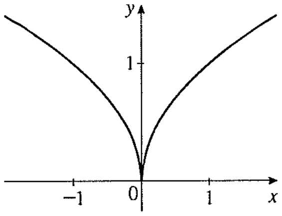
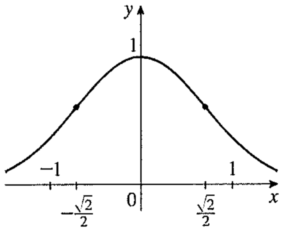


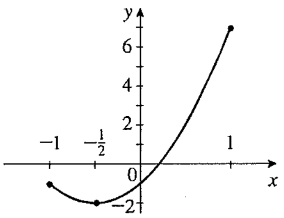
80.



81.



82.



83. **a.** $N'(t)$ is positive because N is increasing on $(0, 12)$.

b. $N''(t) < 0$ on $(0, 6)$ and $N''(t) > 0$ on $(6, 12)$.

c. The rate of growth of the number of help-wanted advertisements was decreasing over the first six months of the year and increasing over the last six months.

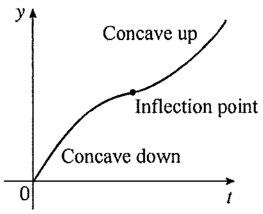
84. **a.** Both $N_1(t)$ and $N_2(t)$ are increasing on $(0, 12)$.

b. $N_1''(t) < 0$ and $N_2''(t) > 0$ on $(0, 12)$.

c. Although the projected number of crimes will increase in either case, a cut in the budget will see an accelerated increase in the number of crimes committed. With the budget intact, the rate of increase of crimes committed will continue to drop.

85. $f(t)$ increases at an increasing rate until the water level reaches the middle of the vase (this corresponds to the inflection point of f). At this point, $f(t)$ is increasing at the fastest rate. Though $f(t)$ still increases until the vase is filled, it does so at a decreasing rate.

86. The behavior of $f(t)$ is just the opposite of that given in the solution to Exercise 85. $f(t)$ increases at a decreasing rate until the water level reaches the middle of the vase (this corresponds to the inflection point of f). After that, $f(t)$ increases until the vase is filled and does so at an increasing rate (see the figure).



87. a. $f'(t) = \frac{d}{dt}\left(0.43t^{0.43}\right) = \left(0.43^2\right)t^{-0.57} = \frac{0.1849}{t^{0.57}}$ is positive if $t \geq 1$. This shows that f is increasing for $t \geq 1$, and this implies that the average state cigarette tax was increasing during the period in question.

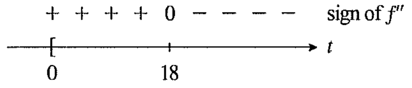
b. $f''(t) = \frac{d}{dt}\left(0.1849t^{-0.57}\right) = (0.1849)(-0.57)t^{-1.57} = -\frac{0.105393}{t^{1.57}}$ is negative if $t \geq 1$. Thus, the rate of the increase of the cigarette tax is decreasing over the period in question.

88. $C'(x) = \frac{d}{dx}\left(1910.5x^{-1.72} + 42.9\right) = -1.72(1910.5)x^{-2.72} = -3286.06x^{-2.72}$, so $C''(x) = -2.72(-3286.06)x^{-2.72} = \frac{8938.0832}{x^{2.72}}$. $C''(x) > 0$ for $x > 0$ and, in particular, for $5 < x < 20$. The per-mile cost of driving a 2012 medium-sized sedan decreases as the number of miles driven increases, but at a decreasing rate.

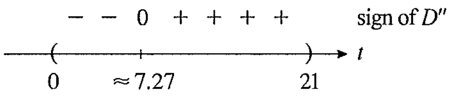
89. a. $A'(t) = \frac{d}{dt}\left(0.012414t^2 + 0.7485t + 313.9\right) = 0.024828t + 0.7485 \geq 0.024828(1) + 0.7485 = 0.773328$, so $A'(t)$ is positive for $t \geq 1$. Therefore, A is increasing on $(1, 56)$, showing that the average amount of atmospheric carbon dioxide is increasing from 1958 through 2013.

b. $A''(t) = \frac{d}{dt}A'(t) = \frac{d}{dt}(0.024828t + 0.7485) = 0.024828 > 0$, showing that the rate of increase of the amount of atmospheric carbon dioxide is increasing from 1958 through 2013.

90. $s = f(t) = -t^3 + 54t^2 + 480t + 6$, so the velocity of the rocket is $v = f'(t) = -3t^2 + 108t + 480$ and its acceleration is $a = f''(t) = -6t + 108 = -6(t - 18)$. From the sign diagram, we see that $(18, 20310)$ is an inflection point of f. Our computations reveal that the maximum velocity of the rocket is attained when $t = 18$. The maximum velocity is $f'(18) = -3(18)^2 + 108(18) + 480 = 1452$, or 1452 ft/sec.



91. a. $D(t) = 0.0032t^3 - 0.0698t^2 + 0.6048t + 3.22$, $0 \leq t \leq 21$, so $D'(t) = 0.0096t^2 - 0.1396t + 0.6048$ and $D''(t) = 0.0192t - 0.1396$. Setting $D''(t) = 0$ gives $t \approx 7.27$. From the sign diagram for D'', we see that the graph of D is concave downward on approximately $(0, 7.27)$ and concave upward on approximately $(7.27, 21)$.



b. The inflection point of the graph of D is approximately $(7.27, 5.16)$. The U.S. public debt was increasing at a decreasing rate from 1990 through the first quarter of 1997 (approximately), and then continued to increase, but at an increasing rate, from that point onward.

Note: In Exercise 4.1.85, we showed that D is increasing on $(0, 21)$.

92. $f(t) = -0.0004401t^3 + 0.007t^2 + 0.112t + 0.28$, so $f'(t) = -0.0013203t^2 + 0.014t + 0.12$ and $f''(t) = -0.0026406t + 0.014$. Setting $f''(t) = 0$, we find $t \approx 5.302$. We see that $f''(t) > 0$ on approximately $(0, 5.3)$ and $f''(t) < 0$ on approximately $(5.3, 21)$, so $t \approx 5.3$ gives an inflection point of f. $f(5.3) \approx -0.0004401(5.3)^3 + 0.007(5.3)^2 + 0.112(5.3) + 0.28 \approx 1.00$, so the point of inflection is approximately $(5.3, 1)$. We conclude that the death rate from AIDS worldwide was increasing most rapidly around March of 1995. At that point the rate was approximately 1 million deaths per year.

93. **a.** $f(t) = -0.083t^3 + 0.6t^2 + 0.18t + 20.1$, so at the beginning of 1960, the median age of women at first marriage was $f(0) = 20.1$. In 2000, it was $f(4) = -0.083(4)^3 + 0.6(4)^2 + 0.18(4) + 20.1 \approx 25.1$, and in 2001, it was $f(5) = -0.083(5)^3 + 0.6(5)^2 + 0.18(5) + 20.1 \approx 25.6$.

b. $f'(t) = -0.083(3t^2) + 0.6(2t) + 0.18 = -0.249t^2 + 1.2t + 0.18$, so $f''(t) = -0.249(2)t + 1.2 = 1.2 - 0.498t$. Thus, $f''(t) = 0$ when $t = \dfrac{1.2}{0.498} \approx 2.41$. Therefore, the median age was changing most rapidly approximately 2.41 decades after the beginning of 1960; that is, early in 1984.

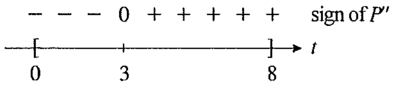
94. **a.** $R'(x) = \dfrac{d}{dx}\left(-0.003x^3 + 1.35x^2 + 2x + 8000\right) = -0.009x^2 + 2.7x + 2$; $R''(x) = -0.018x + 2.7$. Setting $R''(x) = 0$ gives $x = 150$. Because $R''(x) > 0$ if $x < 150$ and $R''(x) < 0$ if $x > 150$, we see that the graph of R is concave upward on $(0, 150)$ and concave downward on $(150, 400)$, so $x = 150$ gives an inflection point of R. Thus, the inflection point is $(150, 28550)$.

b. $R''(140) = 0.18$ and $R''(160) = -0.18$ This shows that at $x = 140$, a slight increase in x (spending) results in increased revenue. At $x = 160$, the opposite conclusion holds. So it would be more beneficial to increase the expenditure when it is \$140,000 than when it is \$160,000.

95. **a.** $A'(t) = \dfrac{d}{dt}\left[0.92(t+1)^{0.61}\right] = 0.92(0.61)(t+1)^{-0.39} = \dfrac{0.5612}{(t+1)^{0.39}} > 0$ on $(0, 4)$, so A is increasing on $(0, 4)$. This tells us that the spending is increasing over the years in question.

b. $A''(t) = (0.5612)(-0.39)(t+1)^{-1.39} = -\dfrac{0.218868}{(t+1)^{1.39}} < 0$ on $(0, 4)$, so A'' is concave downward on $(0, 4)$. This tells us that the spending is increasing but at a decreasing rate.

96. $P(t) = t^3 - 9t^2 + 40t + 50$, so $P'(t) = 3t^2 - 18t + 40$ and $P''(t) = 6t - 18 = 6(t - 3)$. The sign diagram of P'' shows that $(3, 116)$ is an inflection point. This analysis reveals that after declining the first 3 years, the growth rate of the company's profit is once again rising.



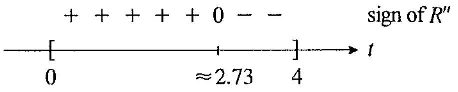
97. **a.** $P(t) = -0.007333t^3 + 0.91343t^2 + 8.507t + 439$, so $P'(t) = -0.021999t^2 + 1.82686t + 8.507$. Setting $P'(t) = 0$ and using the quadratic formula, we find $t = \dfrac{-1.82686 \pm \sqrt{(1.82686)^2 - 4(-0.021999)(8.507)}}{2(-0.021999)} \approx -4.42$ or 87.46. Both roots lie outside $(0, 31)$, so P has no critical number on that interval. Since $P'(1) \approx 10.31 > 0$, we see that $P'(t) > 0$ on $(0, 31)$. We conclude that the number of people aged 80 and over in Canada was increasing from 1981 through 2011.

b. $P''(t) = -0.043998t + 1.82686$. Setting $P''(t) = 0$ and solving, we find $t \approx 41.52$, which lies outside the interval $(0, 31)$. Because $P''(1) = 1.782862 > 0$, we see that $P''(t) > 0$ for all t in $(0, 31)$. Therefore, P' is increasing on $(0, 31)$. We conclude that the population of Canadians aged 80 and over was increasing at an increasing rate from 1981 through 2011.

98. **a.** $R(t) = -0.2t^3 + 1.64t^2 + 1.31t + 3.2$, so $R'(t) = -0.6t^2 + 3.28t + 1.31$ and $R''(t) = -1.2t + 3.28$.

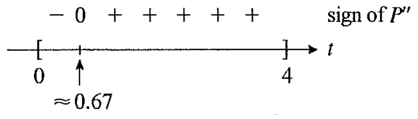
b. Setting $R'(t) = 0$ and solving for t gives $t = \dfrac{-3.28 \pm \sqrt{(3.28)^2 - 4(-0.6)(1.31)}}{2(-0.6)} \approx -0.374$ or 5.840. Both roots lie outside the interval $(0, 4)$. Because $R'(0) = 1.31 > 0$, we conclude that $R'(t) > 0$ for all t in $(0, 4)$.

c. Setting $R''(t) = 0$ gives $-1.2t + 3.28 = 0$, so $t \approx 2.73$. From the sign diagram, we see that R has an inflection point at approximately $(2.73, 14.93)$. This says that between 2004 and 2008, Google's revenue was increasing fastest in late August 2006.



99. a. $P'(t) = \frac{d}{dt}\left(44560t^3 - 89394t^2 + 234633t + 273288\right) = 133680t^2 - 178788 + 234633$. Observe that P' is continuous everywhere and $P'(t) = 0$ has no real solution since the discriminant $b^2 - 4ac = (-178788)^2 - 4(133680)(234633) = -93497808816 < 0$. Because $P'(0) = 234633 > 0$, we may conclude that $P'(t) > 0$ for all t in $(0, 4)$, so the population is always increasing.

b. $P''(t) = 267360t - 178788 = 0$ implies that $t = 0.67$. The sign diagram of P''shows that $t = 0.67$ is an inflection point of the graph of P, so the population was increasing at the slowest pace sometime during August of 1976.



100. a. $D(t) = D_2(t) - D_1(t) = 0.035t^2 + 0.211t + 0.24 - \left(0.0275t^2 + 0.081t + 0.07\right) = 0.0075t^2 + 0.13t + 0.17$, so $D'(t) = 0.015t + 0.13 \geq 0.13$ for $0 < t < 3$, showing that D is increasing on $(0, 3)$. Thus, the projected difference in year t between the deficit without the \$160 million rescue package and the deficit with the rescue package is increasing between 2011 and 2014.

b. $D''(t) = 0.015 > 0$ on $(0, 3)$, and so D is concave upward on $(0, 3)$. This says that the difference referred to in part (a) is increasing at an increasing rate between 2011 and 2014.

101. $A(t) = 1.0974t^3 - 0.0915t^4$, so $A'(t) = 3.2922t^2 - 0.366t^3$ and $A''(t) = 6.5844t - 1.098t^2$. Setting $A'(t) = 0$, we obtain $t^2(3.2922 - 0.366t) = 0$, and this gives $t = 0$ or $t \approx 8.995 \approx 9$. Using the Second Derivative Test, we find $A''(9) = 6.5844(9) - 1.098(81) = -29.6784 < 0$, and this tells us that $t \approx 9$ gives rise to a relative maximum of A. Our analysis tells us that on that May day, the level of ozone peaked at approximately 4 p.m.

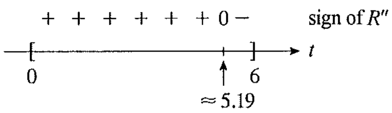
102. a. $N'(t) = \frac{d}{dt}\left(-0.9307t^3 + 74.04t^2 + 46.8667t + 3967\right) = -2.7921t^2 + 148.08t + 46.8667$. $N'$is continuous everywhere and has zeros at $t = \dfrac{-148.08 \pm \sqrt{(148.08)^2 - 4(-0.9307)(46.8667)}}{2(-2.7921)}$, that is, at $t \approx -0.31$ or 53.35. Both these numbers lie outside the interval of interest. Picking $t = 0$ for a test number, we see that $N'(0) = 46.86667 > 0$ and conclude that N is increasing on $(0, 16)$. This shows that the number of participants is increasing over the years in question.

b. $N''(t) = \frac{d}{dt}\left(-2.7921t^2 + 148.08t + 46.86667\right) = -5.5842t + 148.08 = 0$ if $t \approx 26.518$. Thus, $N''(t)$ does not change sign in the interval $(0, 16)$. Because $N''(0) = 148.08 > 0$, we see that $N'(t)$ is increasing on $(0, 16)$ and the desired conclusion follows.

103. a. $R'(t) = \frac{d}{dt}\left(0.00731t^4 - 0.174t^3 + 1.528t^2 + 0.48t + 19.3\right) = 0.02924t^3 - 0.522t^2 + 3.056t + 0.48$ and $R''(t) = 0.08772t^2 - 1.044t + 3.056$. Solving the equation $R''(t) = 0$, we obtain

$$t = \frac{1.044 \pm \sqrt{(-1.044)^2 - 4(0.08772)(3.056)}}{2(0.08772)} \approx 5.19 \text{ or } 6.71.$$

From the sign diagram of R'', we see that the inflection points are approximately $(5.19, 43.95)$ and $(6.71, 53.56)$. We see that the dependency ratio will be increasing at the greatest pace around $t = 5.2$, that is, around 2052.



b. The dependency ratio will be $R(5.2) \approx 43.99$, or approximately 44.

104. True. If f' is increasing on (a, b), then $-f'$ is decreasing on (a, b), and so if the graph of f is concave upward on (a, b), the graph of $-f$ must be concave downward on (a, b).

105. False. Let $f(x) = x + \frac{1}{x}$ (see Example 2). Then f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$, but f does not have an inflection point at 0.

106. False. If $f(x) = x^2 - 1$, then f is concave up on $(0, 1)$, but $g(x) = \left(x^2 - 1\right)^2 = x^4 - 2x^2 + 1$ and $g''(x) = 12x^2 - 4 < 0$ on $\left(0, \frac{\sqrt{3}}{3}\right)$, so g is not concave up on $(0, 1)$.

107. False. Take $f(x) = x^{1/3}$ on $(-1, 1)$. Then f is defined on $(-1, 1)$ and f has an inflection point at $(0, 0)$, but $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}}$, so $f''(0)$ is undefined.

108. True. The given conditions imply that $f''(0) < 0$ and the Second Derivative Test gives the desired conclusion.

109. $f(x) = ax^2 + bx + c$, so $f'(x) = 2ax + b$ and $f''(x) = 2a$. Thus, $f''(x) > 0$ if $a > 0$, and the parabola opens upward. If $a < 0$, then $f''(x) < 0$ and the parabola opens downward.

110. a. $f'(x) = 3x^2$, $g'(x) = 4x^3$, and $h'(x) = -4x^3$. Setting $f'(x) = 0$, $g'(x) = 0$, and $h'(x) = 0$ gives $x = 0$ as a critical number of each function.

b. $f''(x) = 6x$, $g''(x) = 12x^2$, and $h''(x) = -12x^2$, so that $f''(0) = 0$, $g''(0) = 0$, and $h''(0) = 0$. Thus, the Second Derivative Test yields no conclusion in these cases.

c. Because $f'(x) > 0$ for both $x > 0$ and $x < 0$, $f'(x)$ does not change sign as we move across the critical number $x = 0$. Thus, by the First Derivative Test, f has no extremum at 0. Next, $g'(x)$ changes sign from negative to positive as we move across 0, showing that g has a relative minimum there. Finally, we see that $h'(x) > 0$ for $x < 0$ and $h'(x) < 0$ for $x > 0$, so h has a relative maximum at $x = 0$.