4 APPLICATIONS OF THE DERIVATIVE

4.1 Applications of the First Derivative

Concept Questions page 263

1. **a.** f is increasing on I if whenever x_1 and x_2 are in I with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

   b. f is decreasing on I if whenever x_1 and x_2 are in I with $x_1 < x_2$, then $f(x_1) > f(x_2)$.

2. Find all numbers such that $f'(x)$ does not exist or $f'(x)$ is not defined. Using test numbers if necessary, draw the sign diagram for f'. On a subinterval where $f'(x) < 0$, f is decreasing; on a subinterval where $f'(x) > 0$, f is increasing.

3. **a.** f has a relative maximum at $x = a$ if there is an open interval I containing a such that $f(x) \leq f(a)$ for all x in I.

   b. f has a relative minimum at $x = a$ if there is an open interval I containing a such that $f(x) \geq f(a)$ for all x in I.

4. **a.** A critical number of f is a number c in the domain of f such that $f'(c) = 0$ or f' does not exist at c.

   b. If f has a relative extremum at c, then c must be a critical number of f.

5. See page 260 of the text.

Exercises page 264

1. f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

2. f is decreasing on $(-\infty, -1)$, constant on $(-1, 1)$, and increasing on $(1, \infty)$.

3. f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 1)$.

4. f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 0)$ and $(0, 1)$.

5. f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$.

6. f is increasing on $(-1, 0)$ and $(1, \infty)$ and decreasing on $(-\infty, -1)$ and $(0, 1)$.

7. f is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

8. f is increasing on $(-\infty, -1)$ and $(-1, \infty)$.

9. Increasing on $(20.2, 20.6)$ and $(21.7, 21.8)$, constant on $(19.6, 20.2)$ and $(20.6, 21.1)$, and decreasing on $(21.1, 21.7)$ and $(21.8, 22.7)$.

10. f is increasing on the interval $(0, 6)$, constant on the intervals $(6, 9)$ and $(14, 15)$, and decreasing on the interval $(9, 14)$. Beginning at 5 A.M., the amount of power generated increases until 11 A.M. ($t = 6$), when the power generated reaches its peak (100% capacity). The solar panel continues to generate at 100% capacity until 2 P.M. ($t = 9$), at which point the output begins to drop, reaching its lowest point at 7 P.M. From 7 P.M. until 8 A.M., the power generated remains constant at its lowest level.

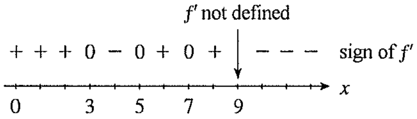
11. **a.** f is decreasing on $(0, 4)$. **b.** f is constant on $(4, 12)$. **c.** f is increasing on $(12, 24)$.

12. **a.** Positive **b.** Positive **c.** Zero **d.** Zero

e. Negative **f.** Negative **g.** Positive

13. **a.** 3, 5, and 7 are critical numbers because $f'(3) = f'(5) = f'(7) = 0$ and 9 is a critical number because $f'(9)$ is not defined.

b.

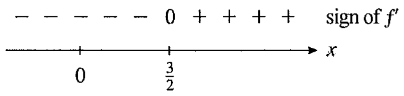


c. f has relative maxima at $(3, 3)$ and $(9, 6)$ and a relative minimum at $(5, 1)$.

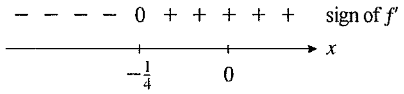
14. $f(x) = 4 - 5x$, so $f'(x) = -5$. Therefore, f is decreasing everywhere; that is, f is decreasing on $(-\infty, \infty)$.

15. $f(x) = 3x + 5$, so $f'(x) = 3 > 0$ for all x. Thus, f is increasing on $(-\infty, \infty)$.

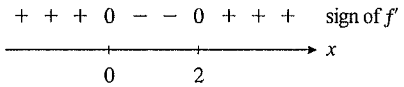
16. $f(x) = x^2 - 3x$, so $f'(x) = 2x - 3$. f' is continuous everywhere and is equal to zero when $x = \frac{3}{2}$. From the sign diagram, we see that f is decreasing on $\left(-\infty, \frac{3}{2}\right)$ and increasing on $\left(\frac{3}{2}, \infty\right)$.



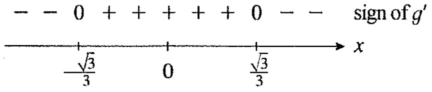
17. $f(x) = 2x^2 + x + 1$, so $f'(x) = 4x + 1 = 0$ if $x = -\frac{1}{4}$. From the sign diagram of f', we see that f is decreasing on $\left(-\infty, -\frac{1}{4}\right)$ and increasing on $\left(-\frac{1}{4}, \infty\right)$.



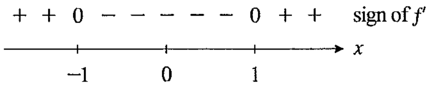
18. $f(x) = x^3 - 3x^2$, so $f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$ if $x = 0$ or 2. From the sign diagram of f', we see that f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.



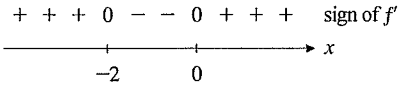
19. $g(x) = x - x^3$, so $g'(x) = 1 - 3x^2$ is continuous everywhere and is equal to zero when $1 - 3x^2 = 0$, or $x = \pm\frac{\sqrt{3}}{3}$. From the sign diagram, we see that f is decreasing on $\left(-\infty, -\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, \infty\right)$ and increasing on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.



20. $f(x) = x^3 - 3x + 4$, so $f'(x) = 3x^2 - 3$. f' is continuous everywhere and is equal to zero when $x = \pm 1$. From the sign diagram, we see that f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$.

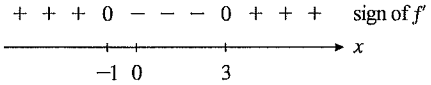


21. $g(x) = x^3 + 3x^2 + 1$, so $g'(x) = 3x^2 + 6x = 3x(x + 2)$. From the sign diagram, we see that g is increasing on $(-\infty, -2)$ and $(0, \infty)$ and decreasing on $(-2, 0)$.

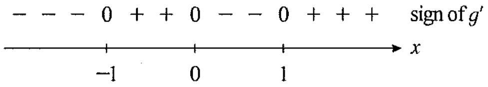


22. $f(x) = \frac{1}{3}x^3 - 3x^2 + 9x + 20$, so $f'(x) = x^2 - 6x + 9 = (x - 3)^2 > 0$ for all x except $x = 3$, at which point $f'(3) = 0$. Therefore, f is increasing on $(-\infty, \infty)$.

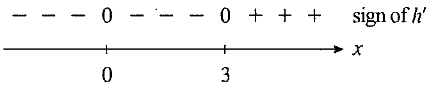
23. $f(x) = \frac{2}{3}x^3 - 2x^2 - 6x - 2$, so $f'(x) = 2x^2 - 4x - 6 = 2\left(x^2 - 2x - 3\right) = 2(x - 3)(x + 1) = 0$ if $x = -1$ or 3. From the sign diagram of f', we see that f is increasing on $(-\infty, -1)$ and $(3, \infty)$ and decreasing on $(-1, 3)$.



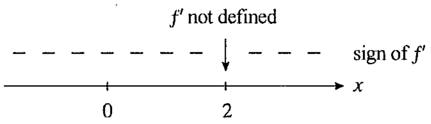
24. $g(x) = x^4 - 2x^2 + 4$, so $g'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right)$ is continuous everywhere and is equal to zero when $x = 0$, 1, or -1. From the sign diagram, we see that g is decreasing on $(-\infty, -1)$ and $(0, 1)$ and increasing on $(-1, 0)$ and $(1, \infty)$.



25. $h(x) = x^4 - 4x^3 + 10$, so $h'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) = 0$ if $x = 0$ or 3. From the sign diagram of h', we see that h is increasing on $(3, \infty)$ and decreasing on $(-\infty, 3)$.

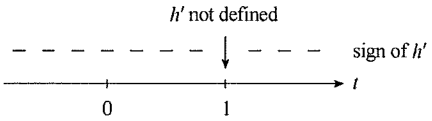


26. $f(x) = \dfrac{1}{x - 2} = (x - 2)^{-1}$, so $f'(x) = -1(x - 2)^{-2}(1) = -\dfrac{1}{(x - 2)^2}$ is discontinuous at $x = 2$ and is continuous and nonzero everywhere else. From the sign diagram, we see that f is decreasing on $(-\infty, 2)$ and $(2, \infty)$.



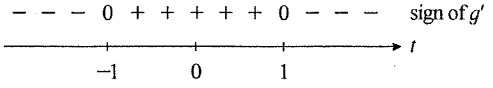
27. $h(x) = \dfrac{1}{2x + 3}$, so $h'(x) = \dfrac{-2}{(2x + 3)^2}$ and we see that h' is not defined at $x = -\frac{3}{2}$. But $h'(x) < 0$ for all x except $x = -\frac{3}{2}$. Therefore, h is decreasing on $\left(-\infty, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, \infty\right)$.

28. $h(t) = \dfrac{t}{t - 1}$, so $h'(t) = \dfrac{(t - 1)(1) - t(1)}{(t - 1)^2} = -\dfrac{1}{(t - 1)^2}$. From the sign diagram, we see that $h'(t) < 0$ whenever it is defined. We conclude that h is decreasing on $(-\infty, 1)$ and $(1, \infty)$.



29. $g(t) = \dfrac{2t}{t^2+1}$, so $g'(t) = \dfrac{(t^2+1)(2)-(2t)(2t)}{(t^2+1)^2} = \dfrac{2t^2+2-4t^2}{(t^2+1)^2} = -\dfrac{2(t^2-1)}{(t^2+1)^2}$. Next, $g'(t) = 0$ if $t = \pm 1$.

From the sign diagram of g', we see that g is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$.



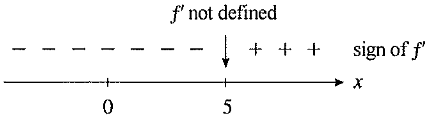
30. $f(x) = x^{3/5}$, so $f'(x) = \frac{3}{5}x^{-2/5} = \dfrac{3}{5x^{2/5}}$. Observe that $f'(x)$ is not defined at $x = 0$, but is positive everywhere else. Therefore, f is increasing on $(-\infty, \infty)$.

31. $f(x) = x^{2/3} + 5$, so $f'(x) = \frac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$, and so f' is not defined at $x = 0$. Now $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$, and so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

32. $f(x) = \sqrt{x+1}$, so $f'(x) = \dfrac{d}{dx}(x+1)^{1/2} = \frac{1}{2}(x+1)^{-1/2} = \dfrac{1}{2\sqrt{x+1}}$, and we see that $f'(x) > 0$ if $x > -1$. Therefore, f is increasing on $(-1, \infty)$.

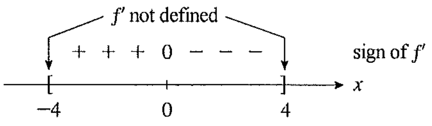
33. $f(x) = (x-5)^{2/3}$, so

$f'(x) = \frac{2}{3}(x-5)^{-1/3} = \dfrac{2}{3(x-5)^{1/3}}$. From the sign diagram, we see that f is decreasing on $(-\infty, 5)$ and increasing on $(5, \infty)$.



34. $f(x) = \sqrt{16-x^2} = \left(16-x^2\right)^{1/2}$, so

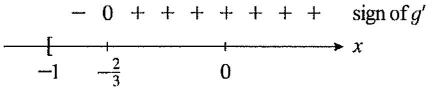
$f'(x) = \frac{1}{2}\left(16-x^2\right)^{-1/2}(-2x) = -\dfrac{x}{\sqrt{16-x^2}}$. Because the domain of f is $[-4, 4]$, we consider the sign diagram for f' on this interval. We see that f is increasing on $(-4, 0)$ and decreasing on $(0, 4)$.



35. $g(x) = x(x+1)^{1/2}$, so

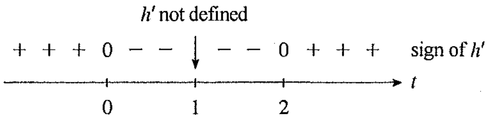
$g'(x) = (x+1)^{1/2} + x\left(\frac{1}{2}\right)(x+1)^{-1/2} = (x+1)^{-1/2}\left(x+1+\frac{1}{2}x\right) = (x+1)^{-1/2}\left(\frac{3}{2}x+1\right) = \dfrac{3x+2}{2\sqrt{x+1}}$.

Thus, g' is continuous on $(-1, \infty)$ and has a zero at $x = -\frac{2}{3}$. From the sign diagram, we see that g is decreasing on $\left(-1, -\frac{2}{3}\right)$ and increasing on $\left(-\frac{2}{3}, \infty\right)$.



36. $f'(x) = \dfrac{d}{dx}\left(x^{-1}-x\right) = -1 - \dfrac{1}{x^2} = -\dfrac{x^2+1}{x^2} < 0$ for all $x \neq 0$. Therefore, f is decreasing on $(-\infty, 0)$ and $(0, \infty)$.

37. $h(x) = \dfrac{x^2}{x-1}$, so $h'(x) = \dfrac{(x-1)(2x) - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$. Thus, h' is continuous everywhere except at $x = 1$ and has zeros at $x = 0$ and $x = 2$. From the sign diagram, we see that h is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 1)$ and $(1, 2)$.



38. f has a relative maximum of $f(0) = 1$ and relative minima of $f(-1) = 0$ and $f(1) = 0$.

39. f has a relative maximum of $f(0) = 1$ and relative minima of $f(-1) = 0$ and $f(1) = 0$.

40. f has a relative maximum at $(0, 0)$ and a relative minimum at $(4, -32)$.

41. f has a relative maximum of $f(-1) = 2$ and a relative minimum of $f(1) = -2$.

42. f has a relative minimum at $(-1, 0)$.

43. f has a relative maximum of $f(1) = 3$ and a relative minimum of $f(2) = 2$.

44. f has a relative minimum at $(0, 2)$.

45. f has a relative maximum at $\left(-3, -\frac{9}{2}\right)$ and a relative minimum at $\left(3, \frac{9}{2}\right)$.

46. f is increasing on the interval (a, c), where $f'(x) > 0$; f is decreasing on (c, d), where $f'(x) < 0$; and f is increasing once again on (d, b), where $f'(x) > 0$.

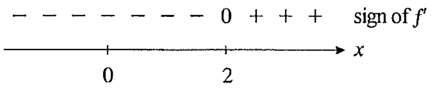
47. f is decreasing on the interval (a, c), where $f'(x) < 0$; f is increasing on (c, d), where $f'(x) > 0$; f is constant on (d, e), where $f'(x) = 0$; and finally f is decreasing on (e, b), where $f'(x) < 0$.

48. The car is moving in the positive direction from $t = a$ to $t = c$, where $v(t) > 0$; it stops at $t = c$, where $v(t) = 0$; then it starts moving in the negative direction from $t = c$ to $t = d$, where $v(t) < 0$. It stops again at $t = d$, where $v(t) = 0$, before moving in the positive direction once again.

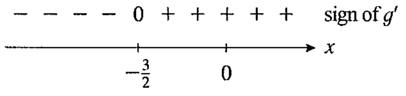
49. The profit is increasing at a level of production between 0 units and c units, corresponding to the interval $(0, c)$ on which $P'(t) > 0$. The profit is neither increasing nor decreasing when the level of production is c units; this corresponds to the number c at which $P'(t) = 0$. Finally, the profit is decreasing when the level of production is between c units and b units.

50. c	**51.** a	**52.** b	**53.** d

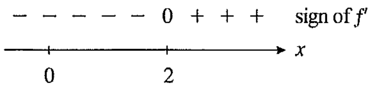
54. $f(x) = x^2 - 4x$, so $f'(x) = 2x - 4 = 2(x-2)$ has a critical point at $x = 2$. From the sign diagram, we see that $f(2) = -4$ is a relative minimum by the First Derivative Test.



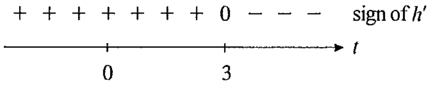
55. $g(x) = x^2 + 3x + 8$, so $g'(x) = 2x + 3$ has a critical point at $x = -\frac{3}{2}$. From the sign diagram, we see that $g\left(-\frac{3}{2}\right) = \frac{23}{4}$ is a relative minimum by the First Derivative Test.



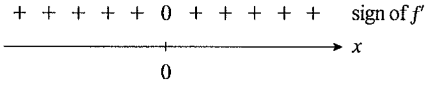
56. $f(x) = \frac{1}{2}x^2 - 2x + 4$, so $f'(x) = x - 2$, giving the critical number $x = 2$. From the sign diagram, we see that $f(2) = 2$ is a relative minimum.



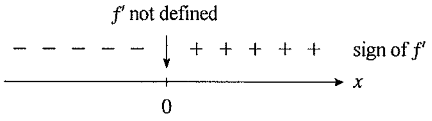
57. $h(t) = -t^2 + 6t + 6$, so $h'(t) = -2t + 6 = -2(t - 3) = 0$ if $t = 3$, a critical number. The sign diagram and the First Derivative Test imply that h has a relative maximum at 3 with value $h(3) = -9 + 18 + 6 = 15$.



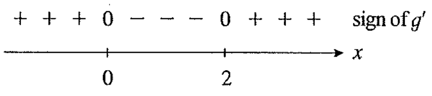
58. $f(x) = x^{5/3}$, so $f'(x) = \frac{5}{3}x^{2/3}$, $x = 0$ as the critical number of f. From the sign diagram, we see that f' does not change sign as we move across $x = 0$, and conclude that f has no relative extremum.



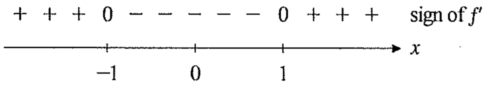
59. $f(x) = x^{2/3} + 2$, so $f'(x) = \frac{2}{3}x^{-1/3} = \dfrac{2}{3x^{1/3}}$ is discontinuous at $x = 0$, a critical number. From the sign diagram and the First Derivative Test, we see that f has a relative minimum at $(0, 2)$.



60. $g(x) = x^3 - 3x^2 + 5$, so $g'(x) = 3x^2 - 6x = 3x(x - 2) = 0$ if $x = 0$ or 2. From the sign diagram, we see that the critical number $x = 0$ gives a relative maximum, whereas $x = 2$ gives a relative minimum. The values are $g(0) = 5$ and $g(2) = 8 - 12 + 5 = 1$.



61. $f(x) = x^3 - 3x + 6$. Setting $f'(x) = 3x^2 - 3 = 3\left(x^2 - 1\right) = 3(x + 1)(x - 1) = 0$ gives $x = -1$ and $x = 1$ as critical numbers. The sign diagram of f' shows that $(-1, 8)$ is a relative maximum and $(1, 4)$ is a relative minimum.



62. $F(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$, so $F'(x) = x^2 - 2x - 3 = (x - 3)(x + 1) = 0$ gives $x = -1$ and $x = 3$ as critical numbers. From the sign diagram, we see that $x = -1$ gives a relative maximum and $x = 3$ gives a relative minimum. The values are $F(-1) = -\frac{1}{3} - 1 + 3 + 4 = \frac{17}{3}$ and $F(3) = 9 - 9 - 9 + 4 = -5$.