66. Differentiating the equation $2h^{1/2} + \frac{1}{25}t - 2\sqrt{20} = 0$ with respect to t gives $2\left(\frac{1}{2}h^{-1/2}\right)\frac{dh}{dt} + \frac{1}{25} = 0$, or $\frac{dh}{dt} = -\frac{\sqrt{h}}{25}$. Therefore, with $h = 8$, we have $\frac{dh}{dt} = \frac{\sqrt{8}}{25} \approx -0.113$. Thus, the height of the water is decreasing at the rate of approximately 0.11 ft/sec.

67. $P^5V^7 = C$, so $V^7 = CP^{-5}$ and $7V^6\dfrac{dV}{dt} = -5CP^{-6}\dfrac{dP}{dt}$. Therefore, $\dfrac{dV}{dt} = -\dfrac{5C}{7P^6V^6}\dfrac{dP}{dt} = -\dfrac{5P^5V^7}{7P^6V^6}\dfrac{dP}{dt} = -\dfrac{5}{7}\dfrac{V}{P}\dfrac{dP}{dt}$. When $V = 4$ L, $P = 100$ kPa, and $\dfrac{dP}{dt} = -5\,\dfrac{\text{kPa}}{\text{sec}}$, we have $\dfrac{dV}{dt} = -\dfrac{5}{7}\cdot\dfrac{4}{100}(-5) = \dfrac{1}{7}\left(\dfrac{\text{L}}{\text{kPa}}\cdot\dfrac{\text{kPa}}{\text{s}}\right) = \dfrac{1}{7}\,\dfrac{\text{L}}{\text{s}}$.

68. When $v = 2.92 \times 10^8$ and $\dfrac{dv}{dt} = a = 2.42 \times 10^5$, $\dfrac{dm}{dt} = \dfrac{(9.11 \times 10^{-31})(2.92 \times 10^8)(2.42 \times 10^5)}{(2.98 \times 10^8)^2\left[1 - \left(\frac{2.92\times10^8}{2.98\times10^8}\right)^2\right]^{3/2}} \approx 9.1 \times 10^{-32}$,

so the mass is increasing at the rate of approximately 9.1×10^{-32} kg/sec.

69. False. There are no real numbers x and y such that $x^2 + y^2 = -1$.

70. True. If $-1 \le x < 0$, then $y^2 = \left(\sqrt{1-x^2}\right)^2 = 1 - x^2$, so $x^2 + y^2 = 1$. If $0 \le x \le 1$, then $y^2 = \left(-\sqrt{1-x^2}\right)^2 = 1 - x^2$, so $x^2 + y^2 = 1$.

71. True. Differentiating both sides of the equation with respect to x, we have $\frac{d}{dx}\left[f(x)g(y)\right] = \frac{d}{dx}(0)$, so $f(x)g'(y)\dfrac{dy}{dx} + f'(x)g(y) = 0$, and therefore $\frac{dy}{dx} = -\frac{f'(x)g(y)}{f(x)g'(y)}$, provided $f(x) \ne 0$ and $g'(y) \ne 0$.

72. True. Differentiating both sides of the equation with respect to x, $\frac{d}{dx}\left[f(x) + g(y)\right] = \frac{d}{dx}(0)$, so $f'(x) + g'(y)\frac{dy}{dx} = 0$, and therefore $\frac{dy}{dx} = -\frac{f'(x)}{g'(y)}$.

73. True. If $y = f(x)$, then $\Delta y = f(x + \Delta x) - f(x) \approx f'(x)\,\Delta x$, from which it follows that $f(x + \Delta x) \approx f(x) + f'(x)\,\Delta x$.

74. True. Let $y = f(x) = x^{1/3}$. Then $y' = f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$. At $x = a$, $\Delta y = f(a + \Delta x) - f(a) \approx f'(a)\,\Delta x$, so $f(a + \Delta x) \approx f(a) + f'(a)\,\Delta x = a^{1/3} + \dfrac{\Delta x}{3a^{2/3}}$. Letting $\Delta x = h$, we have $(a+h)^{1/3} = f(a+h) \approx a^{1/3} + \dfrac{h}{3a^{2/3}}$.

3.7 Differentials

Concept Questions

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1. The differential of x is dx. The differential of y is $dy = f'(x)\,dx$.

2. a. $A = \Delta x$, $B = \Delta y$, and $C = dy$.

b. $f'(x) = \dfrac{dy}{\Delta x}$.

c. From part (b), we see that $dy = f'(x)\,\Delta x$. Because $B \approx C$, $\Delta y \approx f'(x)\,\Delta x = f'(x)\,dx$.

3. Because $\Delta P = P(t_0 + \Delta t) - P(t_0) \approx P'(t_0)\,\Delta t$, we see that $P'(t_0)\,\Delta t$ is an approximation of the change in the population from time t_0 to time $t_0 + \Delta t$.

4. $P(t) = P(t_0 + \Delta t) \approx P(t_0) + P'(t_0)\,\Delta t$.

Exercises

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1. $f(x) = 2x^2$ and $dy = 4x\,dx$.

2. $f(x) = 3x^2 + 1$ and $dy = 6x\,dx$.

3. $f(x) = x^3 - x$ and $dy = \left(3x^2 - 1\right)dx$.

4. $f(x) = 2x^3 + x$ and $dy = \left(6x^2 + 1\right)dx$.

5. $f(x) = \sqrt{x+1} = (x+1)^{1/2}$ and $dy = \frac{1}{2}(x+1)^{-1/2}\,dx = \dfrac{dx}{2\sqrt{x+1}}$.

6. $f(x) = 3x^{-1/2}$ and $dy = -\dfrac{3}{2x^{3/2}}\,dx$.

7. $f(x) = 2x^{3/2} + x^{1/2}$ and $dy = \left(3x^{1/2} + \frac{1}{2}x^{-1/2}\right)dx = \frac{1}{2}x^{-1/2}(6x+1)\,dx = \dfrac{6x+1}{2\sqrt{x}}\,dx$.

8. $f(x) = 3x^{5/6} + 7x^{2/3}$ and $dy = \left(\frac{5}{2}x^{-1/6} + \frac{14}{3}x^{-1/3}\right)dx$.

9. $f(x) = x + \dfrac{2}{x}$ and $dy = \left(1 - \dfrac{2}{x^2}\right)dx = \dfrac{x^2-2}{x^2}dx$.

10. $f(x) = \dfrac{3}{x-1}$ and $dy = -\dfrac{3}{(x-1)^2}dx$.

11. $f(x) = \dfrac{x-1}{x^2+1}$ and $dy = \dfrac{x^2+1-(x-1)\,2x}{\left(x^2+1\right)^2}\,dx = \dfrac{-x^2+2x+1}{\left(x^2+1\right)^2}\,dx$.

12. $f(x) = \dfrac{2x^2+1}{x+1}$ and $dy = \dfrac{(x+1)(4x)-\left(2x^2+1\right)}{(x+1)^2}\,dx = \dfrac{2x^2+4x-1}{(x+1)^2}\,dx$.

13. $f(x) = \sqrt{3x^2 - x} = \left(3x^2 - x\right)^{1/2}$ and $dy = \frac{1}{2}\left(3x^2 - x\right)^{-1/2}(6x-1)\,dx = \dfrac{6x-1}{2\sqrt{3x^2-x}}\,dx$.

14. $f(x) = \left(2x^2+3\right)^{1/3}$ and $dy = \frac{1}{3}\left(2x^2+3\right)^{-2/3}(4x)\,dx = \dfrac{4x}{3\left(2x^2+3\right)^{2/3}}\,dx$.

15. $f(x) = x^2 - 1.$

a. $dy = 2x\,dx.$

b. $dy \approx 2(1)(0.02) = 0.04.$

c. $\Delta y = \left[(1.02)^2 - 1\right] - (1 - 1) = 0.0404.$

16. $f(x) = 3x^2 - 2x + 6$

a. $dy = (6x - 2)\,dx.$

b. $dy \approx 10(-0.03) = -0.3.$

c. $\Delta y = \left[3(1.97)^2 - 2(1.97) + 6\right] - \left[3(2)^2 - 2(2) + 6\right] = -0.2973.$

17. $f(x) = \dfrac{1}{x}.$

a. $dy = -\dfrac{dx}{x^2}.$

b. $dy \approx -0.05.$

c. $\Delta y = \frac{1}{-0.95} - \frac{1}{-1} \approx -0.05263.$

18. $f(x) = \sqrt{2x+1} = (2x+1)^{1/2}.$

a. $dy = \frac{1}{2}(2x+1)^{-1/2}(2)\,dx = \dfrac{dx}{\sqrt{2x+1}}.$

b. $dy \approx \frac{0.1}{\sqrt{9}} \approx 0.03333.$

c. $\Delta y = [2(4.1)+1]^{1/2} - [2(4)+1]^{1/2} \approx 0.03315.$

19. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{10} \approx 3 + \dfrac{1}{2 \cdot \sqrt{9}} \approx 3.167.$

20. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{17} \approx 4 + \dfrac{1}{2 \cdot 4} = 4.125.$

21. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{49.5} \approx 7 + \dfrac{0.5}{2 \cdot 7} \approx 7.0357.$

22. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{99.7} \approx 10 - \dfrac{0.3}{2 \cdot 10} = 9.985.$

23. $y = x^{1/3}$ and $dy = \frac{1}{3}x^{-2/3}\,dx$. Therefore, $\sqrt[3]{7.8} \approx 2 - \dfrac{0.2}{3 \cdot 4} \approx 1.983.$

24. $y = x^{1/4}$ and $dy = \frac{1}{4}x^{-3/4}\,dx$. Therefore, $\sqrt[4]{81.6} \approx 3 + \dfrac{0.6}{4 \cdot 27} \approx 3.0056.$

25. $y = \sqrt{x}$ and $dy = \dfrac{dx}{2\sqrt{x}}$. Therefore, $\sqrt{0.089} = \frac{1}{10}\sqrt{8.9} \approx \frac{1}{10}\left(3 - \frac{0.1}{2 \cdot 3}\right) \approx 0.298.$

26. $y = \sqrt[3]{x}$ and $dy = \dfrac{dx}{3x^{2/3}}$. Therefore, $\sqrt[3]{0.00096} = \dfrac{1}{100}\sqrt[3]{960} \approx \dfrac{1}{100}\left[10 - \dfrac{40}{3(100)}\right] \approx 0.0987.$

27. $y = f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$. Therefore, $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} - \dfrac{1}{2}x^{-3/2}$, so $dy = \left(\dfrac{1}{2x^{1/2}} - \dfrac{1}{2x^{3/2}}\right)dx.$ Letting $x = 4$ and $dx = 0.02$, we find $\sqrt{4.02} + \dfrac{1}{\sqrt{4.02}} - f(4) = f(4.02) - f(4) = \Delta y \approx dy$, so

$$\sqrt{4.02} + \frac{1}{\sqrt{4.02}} \approx f(4) + dy \approx 2 + \frac{1}{2} + \left(\frac{1}{2 \cdot 2} - \frac{1}{16}\right)(0.02) = 2.50375.$$

28. Let $y = f(x) = \dfrac{2x}{x^2+1}$. Then $\dfrac{dy}{dx} = \dfrac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$ and $dy = \dfrac{2(1-x^2)}{(x^2+1)^2}\,dx$.

Letting $x = 5$ and $dx = -0.02$, we find $f(5) - f(4.98) = \dfrac{2(5)}{5^2+1} - \dfrac{2(4.98)}{(4.98)^2+1} = \Delta y \approx dy$, so

$\dfrac{2(4.98)}{(4.98)^2+1} \approx \dfrac{10}{26} - \dfrac{2(1-5^2)}{(5^2+1)^2}(-0.02) \approx 0.3832.$

29. The volume of the cube is given by $V = x^3$. Then $dV = 3x^2\,dx$, and when $x = 12$ and $dx = 0.02$, $dV = 3(144)(\pm 0.02) = \pm\,8.64$. The possible error that might occur in calculating the volume is $\pm 8.64\ \text{cm}^3$.

30. The area of the cube of side x cm is $S = 6x^2$. Thus, the amount of paint required is approximately $\Delta S = 6(x+\Delta x)^2 - 6x^2 \approx dS = 12x\,dx$. With $x = 30$ and $dx = \Delta x = 0.05$, $\Delta S \approx 12(30)(0.05) = 18$, or approximately $18\ \text{cm}^3$.

31. The volume of the hemisphere is given by $V = \frac{2}{3}\pi r^3$. The amount of rust-proofer needed is $\Delta V = \frac{2}{3}\pi(r+\Delta r)^3 - \frac{2}{3}\pi r^3 \approx dV = \frac{2}{3}(3\pi r^2)\,dr$. Thus, with $r = 60$ and $dr = \dfrac{1}{12}(0.01)$, we have $\Delta V \approx 2\pi(60^2)\left(\frac{1}{12}\right)(0.01) \approx 18.85$. So we need approximately $18.85\ \text{ft}^3$ of rust-proofer.

32. The volume of the tumor is given by $V = \frac{4}{3}\pi r^3$. Then $dV = 4\pi r^2\,dr$. When $r = 1.1$ and $dr = 0.005$, $dV = 4\pi(1.1)^2(\pm 0.005) = \pm 0.076\ \text{cm}^3$.

33. $dR = \dfrac{d}{dr}\left(k\ell r^{-4}\right)dr = -4k\ell r^{-5}\,dr$. With $\dfrac{dr}{r} = 0.1$, we find $\dfrac{dR}{R} = -\dfrac{4k\ell r^{-5}}{k\ell r^{-4}}\,dr = -4\dfrac{dr}{r} = -4(0.1) = -0.4$. In other words, the resistance will drop by 40%.

34. $f(x) = 640x^{1/5}$ and $df = 128x^{-4/5}\,dx$. When $x = 243$ and $dx = 5$, we have $df = 128(243)^{-4/5}(5) = 128\left(\frac{5}{81}\right) \approx 7.9$, or approximately \$7.9 billion.

35. $f(n) = 4n\sqrt{n-4} = 4n(n-4)^{1/2}$, so $df = 4\left[(n-4)^{1/2} + \frac{1}{2}n(n-4)^{-1/2}\right]dn$. When $n = 85$ and $dn = 5$, $df = 4\left(9 + \dfrac{85}{2\cdot 9}\right)5 \approx 274$ seconds.

36. $P(x) = -\frac{1}{8}x^2 + 7x + 30$ and $dP = \left(-\frac{1}{4}x + 7\right)dx$. To estimate the increase in profits when the amount spent on advertising each quarter is increased from \$24,000 to \$26,000, we set $x = 24$ and $dx = 2$ and compute $dP = \left(-\frac{24}{4} + 7\right)(2) = 2$, or \$2000.

37. $N(r) = \dfrac{7}{1+0.02r^2}$ and $dN = -\dfrac{0.28r}{\left(1+0.02r^2\right)^2}\,dr$. To estimate the decrease in the number of housing starts when the mortgage rate is increased from 6% to 6.5%, we set $r = 6$ and $dr = 0.5$ and compute $dN = -\dfrac{(0.28)(6)(0.5)}{(1.72)^2} \approx -0.283937$, or 283,937 fewer housing starts.

38. $s(x) = 0.3\sqrt{x} + 10$ and $s' = \frac{0.15}{\sqrt{x}}\,dx$. To estimate the change in price when the quantity supplied is increased from 10,000 units to 10,500 units, we compute $ds = \frac{(0.15)\,500}{100} = 0.75$, or 75 cents.

39. $p = \frac{30}{0.02x^2 + 1}$ and $dp = -\frac{1.2x}{\left(0.02x^2 + 1\right)^2}\,dx$. To estimate the change in the price p when the quantity demanded changed from 5000 to 5500 units per week (that is, x changes from 5 to 5.5), we compute $dp = \frac{(-1.2)\,(5)\,(0.5)}{[0.02\,(25) + 1]^2} \approx -1.33$, a decrease of \$1.33.

40. $S = kW^{2/3}$ and $dS = \frac{0.2}{3W^{1/3}}\,dW$. To determine the percentage error in the calculation of the surface area of a horse that weighs 300 kg when the maximum error in measurement is 0.6 kg and $k = 0.1$, we compute $\frac{dS}{S} = \frac{0.2}{3W^{1/3}}\,dW \cdot \frac{1}{0.1W^{2/3}} = \frac{2}{3W}\,dW = \frac{2\,(0.6)}{3\,(300)} \approx 0.00133$, or 0.133%.

41. $P(x) = -0.000032x^3 + 6x - 100$ and $dP = \left(-0.000096x^2 + 6\right)dx$. To determine the error in the estimate of Trappee's profits corresponding to a maximum error in the forecast of 15 percent [that is, $dx = \pm 0.15\,(200)$], we compute $dP = \left[(-0.000096)\,(200)^2 + 6\right](\pm 30) = (2.16)\,(30) = \pm 64.80$, or \$64,800.

42. $p = \frac{55}{2x^2 + 1}$ and $dp = -\frac{220x}{\left(2x^2 + 1\right)^2}\,dx$. To find the error corresponding to a possible error of 15% in a forecast of 1.8 billion bushels, we compute $dp = -\frac{(220)\,(1.8)\,(\pm 0.27)}{\left[2\,(1.8)^2 + 1\right]^2} \approx \pm 1.91$, or approximately \$1.91/bushel.

43. The approximate change in the quantity demanded is given by $\Delta x \approx dx = f'(p)\,\Delta p = \frac{d}{dp}(144 - p)^{1/2}\,\Delta p = -\frac{1}{2} \cdot \frac{1}{\sqrt{144 - p}} \cdot \Delta p$. When $\Delta p = 110 - 108 = 2$, we find $\Delta x = -\frac{1}{2} \cdot \frac{1}{\sqrt{144 - 108}}\,(2) = -\frac{1}{6} \approx -0.1667$. Thus, the quantity demanded decreases by approximately 167 tires/week.

44. The change is given by

$$\Delta A \approx dA = A'(t)\,dt = 136\frac{d}{dt}\left\{\left[1 + 0.25\,(t - 4.5)^2\right]^{-1} + 28\right\}\Delta t$$
$$= 136\left[1 + 0.25\,(t - 4.5)^2\right]^{-2}(0.25)\,(2)\,(t - 4.5)\,\Delta t = \frac{68\,(t - 4.5)}{\left[1 + 0.25\,(t - 4.5)^2\right]^2}\,\Delta t.$$

When $t = 8$ and $\Delta t = 8.05 - 8 = 0.05$, we find $A \approx 0.7210$, so the change in the amount of nitrogen dioxide is approximately 0.72 PSI.

45. $N(x) = \dfrac{500(400+20x)^{1/2}}{(5+0.2x)^2}$ and

$N'(x) = \dfrac{(5+0.2x)^2\, 250(400+20x)^{-1/2}(20) - 500(400+20x)^{1/2}(2)(5+0.2x)(0.2)}{(5+0.2x)^4}\, dx$. To estimate the change in the number of crimes if the level of reinvestment changes from 20 cents to 22 cents per dollar deposited, we compute

$$dN = \frac{(5+4)^2(250)(800)^{-1/2}(20) - 500(400+400)^{1/2}(2)(9)(0.2)}{(5+4)^4}(2) \approx \frac{(14318.91 - 50911.69)}{9^4}(2)$$

≈ -11, a decrease of approximately 11 crimes per year.

46. a. $P = \dfrac{20{,}000r}{1-\left(1+\frac{r}{12}\right)^{-360}}$ and

$$dP = \frac{\left[1-\left(1+\frac{r}{12}\right)^{-360}\right]20{,}000 - 20{,}000r(360)\left(1+\frac{r}{12}\right)^{-361}\left(\frac{1}{12}\right)}{\left[1-\left(1+\frac{r}{12}\right)^{-360}\right]^2}dr$$
$$= \frac{20{,}000\left\{\left[1-\left(1+\frac{r}{12}\right)^{-360}\right] - 30r\left(1+\frac{r}{12}\right)^{-361}\right\}}{\left[1-\left(1+\frac{r}{12}\right)^{-360}\right]^2}dr$$

b. When $r = 0.05$, $dP \approx \dfrac{20{,}000(0.776173404 - 0.334346782)}{(0.776173404)^2} \approx 14{,}667.77912\, dr$. When the interest rate increases from 5% to 5.2% per year, $dP = 14{,}667.77912(0.002) \approx 29.34$, or approximately \$29.34. When the interest rate increases from 5% to 5.3% per year, $dP = 14{,}667.77912(0.003) \approx 44.00$, or approximately \$44.00. When the interest rate increases from 5% to 5.4% per year, $dP = 14{,}667.77912(0.004) \approx 58.67$, or approximately \$58.67. When the interest rate increases from 5% to 5.5% per year, $dP = 14{,}667.77912(0.005) \approx 73.34$, or approximately \$73.34.

47. $A = 10{,}000\left(1+\frac{r}{12}\right)^{120}$.

a. $dA = 10{,}000(120)\left(1+\frac{r}{12}\right)^{119}\left(\frac{1}{12}\right)dr = 100{,}000\left(1+\frac{r}{12}\right)^{119}dr$.

b. At 3.1%, it will be worth $100{,}000\left(1+\frac{0.03}{12}\right)^{119}(0.001)$, or approximately \$134.60 more. At 3.2%, it will be worth $100{,}000\left(1+\frac{0.03}{12}\right)^{119}(0.002)$, or approximately \$269.20 more. At 3.3%, it will be worth $100{,}000\left(1+\frac{0.03}{12}\right)^{119}(0.003)$, or approximately \$403.80 more.

48. $S = \dfrac{24{,}000\left[\left(1+\frac{r}{12}\right)^{300} - 1\right]}{r}$

a. $dS = 24{,}000\left[\dfrac{(r)300\left(1+\frac{r}{12}\right)^{299}\left(\frac{1}{12}\right) - \left(1+\frac{r}{12}\right)^{300} + 1}{r^2}\right]dr$.

b. With $r = 0.04$, we find $dS = 14{,}864{,}762.53\, dr$. Therefore, if John's account earned 4.1%, it would be worth $dS = 14{,}864{,}762.53(0.001) \approx \$14{,}864.76$ more; if it earned 4.2%, it would be worth $dS = 14{,}864{,}762.53(0.002) \approx \$29{,}729.53$ more, and if it earned 4.3%, it would be worth $dS = 14{,}864{,}762.53(0.003) \approx \$44{,}594.29$ more.