3.6 Implicit Differentiation and Related Rates

Concept Questions page 230

1. a. We differentiate both sides of $F(x, y) = 0$ with respect to x, then solve for dy/dx.
   b. The Chain Rule is used to differentiate any expression involving the dependent variable y.

2. $xg(y) + yf(x) = 0$. Differentiating both sides with respect to x gives $xg'(y)y' + g(y) + y'f(x) + yf'(x) = 0$, so $[xg'(y) + f(x)]y' = -[g(y) + yf'(x)]$, and finally $y' = -\dfrac{g(y) + yf'(x)}{f(x) + xg'(y)}$

3. Suppose x and y are two variables that are related by an equation. Furthermore, suppose x and y are both functions of a third variable t. (Normally t represents time.) Then a related rates problem involves finding dx/dt or dy/dt.

4. See page 228 in the text.

Exercises page 231

1. a. Solving for y in terms of x, we have $y = -\frac{1}{2}x + \frac{5}{2}$. Therefore, $y' = -\frac{1}{2}$.
   b. Next, differentiating $x + 2y = 5$ implicitly, we have $1 + 2y' = 0$, or $y' = -\frac{1}{2}$.

2. a. Solving for y in terms of x, we have $y = -\frac{3}{4}x + \frac{3}{2}$. Therefore, $y' = -\frac{3}{4}$.
   b. Next, differentiating $3x + 4y = 6$ implicitly, we obtain $3 + 4y' = 0$, or $y' = -\frac{3}{4}$.

3. a. $xy = 1$, $y = \dfrac{1}{x}$, and $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$.
   b. $x\dfrac{dy}{dx} + y = 0$, so $x\dfrac{dy}{dx} = -y$ and $\dfrac{dy}{dx} = -\dfrac{y}{x} = \dfrac{-1/x}{x} = -\dfrac{1}{x^2}$.

4. a. Solving for y, we have $y(x-1) = 1$ or $y = (x-1)^{-1}$. Therefore, $y' = -(x-1)^{-2} = -\dfrac{1}{(x-1)^2}$.
   b. Next, differentiating $xy - y - 1 = 0$ implicitly, we obtain $y + xy' - y' = 0$, or $y'(x-1) = -y$, so $y' = -\dfrac{y}{x-1} = -\dfrac{1}{(x-1)^2}$.

5. $x^3 - x^2 - xy = 4$.
   a. $-xy = 4 - x^3 + x^2$, so $y = -\dfrac{4}{x} + x^2 - x$ and $\dfrac{dy}{dx} = \dfrac{4}{x^2} + 2x - 1$.
   b. $x^3 - x^2 - xy = 4$, so $-x\dfrac{dy}{dx} = -3x^2 + 2x + y$, and therefore
   $$\frac{dy}{dx} = 3x - 2 - \frac{y}{x} = 3x - 2 - \frac{1}{x}\left(-\frac{4}{x} + x^2 - x\right) = 3x - 2 + \frac{4}{x^2} - x + 1 = \frac{4}{x^2} + 2x - 1.$$

6. $x^2y - x^2 + y - 1 = 0$.
   a. $(x^2 + 1)y = 1 + x^2$, or $y = \dfrac{1 + x^2}{1 + x^2} = 1$. Therefore, $\dfrac{dy}{dx} = 0$.
   b. Differentiating implicitly, $x^2y' + 2xy - 2x + y' = 0$, so $(x^2 + 1)y' = 2x(1 - y)$, and thus $y' = \dfrac{2x(1-y)}{x^2+1}$.
   But from part (a), we know that $y = 1$, so $y' = \dfrac{2x(1-1)}{x^2+1} = 0$.

7. a. $\frac{x}{y} - x^2 = 1$ is equivalent to $\frac{x}{y} = x^2 + 1$, or $y = \frac{x}{x^2+1}$. Therefore, $y' = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$.

b. Next, differentiating the equation $x - x^2y = y$ implicitly, we obtain $1 - 2xy - x^2y' = y'$, $y'(1+x^2) = 1 - 2xy$, and thus $y' = \frac{1-2xy}{(1+x^2)}$. This may also be written in the form $-2y^2 + \frac{y}{x}$. To show that this is equivalent to the results obtained earlier, use the earlier value of y to get $y' = \frac{1 - 2x\left(\frac{x}{x^2+1}\right)}{1+x^2} = \frac{x^2+1-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$.

8. a. $\frac{y}{x} - 2x^3 = 4$ is equivalent to $y = 2x^4 + 4x$. Therefore, $y' = 8x^3 + 4$.

b. Next, differentiating the equation $y - 2x^4 = 4x$ implicitly, we obtain $y' - 8x^3 = 4$, and so $y' = 8x^3 + 4$, as obtained earlier.

9. $x^2 + y^2 = 16$. Differentiating both sides of the equation implicitly, we obtain $2x + 2yy' = 0$, and so $y' = -x/y$.

10. $2x^2 + y^2 = 16$, $4x + 2y\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = -\frac{2x}{y}$.

11. $x^2 - 2y^2 = 16$. Differentiating implicitly with respect to x, we have $2x - 4y\frac{dy}{dx} = 0$, and so $\frac{dy}{dx} = \frac{x}{2y}$.

12. $x^3 + y^3 + y - 4 = 0$. Differentiating both sides of the equation implicitly, we obtain $3x^2 + 3y^2y' + y' = 0$ or $y'(3y^2+1) = -3x^2$. Therefore, $y' = -\frac{3x^2}{3y^2+1}$.

13. $x^2 - 2xy = 6$. Differentiating both sides of the equation implicitly, we obtain $2x - 2y - 2xy' = 0$ and so $y' = \frac{x-y}{x} = 1 - \frac{y}{x}$.

14. $x^2 + 5xy + y^2 = 10$. Differentiating both sides of the equation implicitly, we obtain $2x + 5y + 5xy' + 2yy' = 0$, $2x + 5y + y'(5x+2y) = 0$, and so $y' = -\frac{2x+5y}{5x+2y}$.

15. $x^2y^2 - xy = 8$. Differentiating both sides of the equation implicitly, we obtain $2xy^2 + 2x^2yy' - y - xy' = 0$, $2xy^2 - y + y'(2x^2y - x) = 0$, and so $y' = \frac{y(1-2xy)}{x(2xy-1)} = -\frac{y}{x}$.

16. $x^2y^3 - 2xy^2 = 5$. Differentiating both sides of the equation implicitly, we obtain $2xy^3 + 3x^2y^2y' - 2y^2 - 4xyy' = 0$, $2y^2(xy-1) + xy(3xy-4)y' = 0$, and so $y' = \frac{2y(1-xy)}{x(3xy-4)}$.

17. $x^{1/2} + y^{1/2} = 1$. Differentiating implicitly with respect to x, we have $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$. Therefore, $\frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}$.

18. $x^{1/3} + y^{1/3} = 1$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$, so $y' = -\frac{x^{-2/3}}{y^{-2/3}} = -\frac{y^{2/3}}{x^{2/3}} = -\left(\frac{y}{x}\right)^{2/3}$.

19. $\sqrt{x+y}=x$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{2}(x+y)^{-1/2}(1+y')=1$, $1+y'=2(x+y)^{1/2}$, and so $y'=2\sqrt{x+y}-1$.

20. $(2x+3y)^{1/3}=x^2$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{3}(2x+3y)^{-2/3}(2+3y')=2x$, $2+3y'=6x(2x+3y)^{2/3}$, and so $y'=\frac{2}{3}\left[3x(2x+3y)^{2/3}-1\right]$.

21. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=1$. Differentiating both sides of the equation implicitly, we obtain $-\dfrac{2}{x^3}-\dfrac{2}{y^3}y'=0$, or $y'=-\dfrac{y^3}{x^3}$.

22. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=5$. Differentiating both sides of the equation implicitly, we obtain $-\dfrac{3}{x^4}-\dfrac{3}{y^4}y'=0$, or $y'=-\dfrac{y^4}{x^4}$.

23. $\sqrt{xy}=x+y$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{2}(xy)^{-1/2}(xy'+y)=1+y'$, so $xy'+y=2\sqrt{xy}(1+y')$, $y'(x-2\sqrt{xy})=2\sqrt{xy}-y$, and so $y'=-\dfrac{2\sqrt{xy}-y}{2\sqrt{xy}-x}=\dfrac{2\sqrt{xy}-y}{x-2\sqrt{xy}}$.

24. $\sqrt{xy}=2x+y^2$. Differentiating both sides of the equation implicitly, we obtain $\frac{1}{2}(xy)^{-1/2}(xy'+y)=2+2yy'$, $xy'+y=4\sqrt{xy}+4\sqrt{xy}yy'$, $y'(x-4y\sqrt{xy})=4\sqrt{xy}-y$, and so $y'=\dfrac{4\sqrt{xy}-y}{x-4y\sqrt{xy}}$.

25. $\dfrac{x+y}{x-y}=3x$, or $x+y=3x^2-3xy$. Differentiating both sides of the equation implicitly, we obtain $1+y'=6x-3xy'-3y$, so $y'+3xy'=6x-3y-1$ and $y'=\dfrac{6x-3y-1}{3x+1}$.

26. $\dfrac{x-y}{2x+3y}=2x$, or $x-y=4x^2+6xy$. Differentiating both sides of the equation implicitly, we have $1-y'=8x+6y+6xy'$, so $y'+6xy'=-8x-6y+1$ and $y'=-\dfrac{8x+6y-1}{6x+1}$.

27. $xy^{3/2}=x^2+y^2$. Differentiating implicitly with respect to x, we obtain $y^{3/2}+x\left(\frac{3}{2}\right)y^{1/2}\dfrac{dy}{dx}=2x+2y\dfrac{dy}{dx}$. Multiply both sides by 2 to get $2y^{3/2}+3xy^{1/2}\dfrac{dy}{dx}=4x+4y\dfrac{dy}{dx}$. Then $(3xy^{1/2}-4y)\dfrac{dy}{dx}=4x-2y^{3/2}$, so $\dfrac{dy}{dx}=\dfrac{2(2x-y^{3/2})}{3xy^{1/2}-4y}$.

28. $x^2y^{1/2}=x+2y^3$. Differentiating implicitly with respect to x, we have $2xy^{1/2}+\frac{1}{2}x^2y^{-1/2}y'=1+6y^2y'$, so $4xy+x^2y'=2y^{1/2}+12y^{5/2}y'$, $y'(x^2-12y^{5/2})=-4xy+2y^{1/2}$, and $y'=\dfrac{2\sqrt{y}-4xy}{x^2-12y^{5/2}}$.

29. $(x+y)^3+x^3+y^3=0$. Differentiating implicitly with respect to x, we obtain $3(x+y)^2\left(1+\dfrac{dy}{dx}\right)+3x^2+3y^2\dfrac{dy}{dx}=0$, $(x+y)^2+(x+y)^2\dfrac{dy}{dx}+x^2+y^2\dfrac{dy}{dx}=0$, $\left[(x+y)^2+y^2\right]\dfrac{dy}{dx}=-\left[(x+y)^2+x^2\right]$, and thus $\dfrac{dy}{dx}=-\dfrac{2x^2+2xy+y^2}{x^2+2xy+2y^2}$.

30. $\left(x+y^2\right)^{10}=x^2+25$. Differentiating both sides of this equation with respect to x, we obtain $10\left(x+y^2\right)^9\left(1+2yy'\right)=2x$, so $1+2yy'=\dfrac{2x}{10\left(x+y^2\right)^9}$, $2yy'=\dfrac{2x}{10\left(x+y^2\right)^9}-1$, and $y'=\dfrac{x-5\left(x+y^2\right)^9}{10y\left(x+y^2\right)^9}$.

31. $4x^2+9y^2=36$. Differentiating the equation implicitly, we obtain $8x+18yy'=0$. At the point $(0,2)$, we have $0+36y'=0$, and the slope of the tangent line is 0. Therefore, an equation of the tangent line is $y=2$.

32. $y^2-x^2=16$. Differentiating both sides of this equation implicitly, we obtain $2yy'-2x=0$. At the point $\left(2,2\sqrt{5}\right)$, we have $4\sqrt{5}y'-4=0$, or $y'=m=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$. Using the point-slope form of an equation of a line, we have $y=\frac{\sqrt{5}}{5}x+\frac{8\sqrt{5}}{5}$.

33. $x^2y^3-y^2+xy-1=0$. Differentiating implicitly with respect to x, we have $2xy^3+3x^2y^2\dfrac{dy}{dx}-2y\dfrac{dy}{dx}+y+x\dfrac{dy}{dx}=0$. At $(1,1)$, $2+3\dfrac{dy}{dx}-2\dfrac{dy}{dx}+1+\dfrac{dy}{dx}=0$, and so $2\dfrac{dy}{dx}=-3$ and $\dfrac{dy}{dx}=-\dfrac{3}{2}$. Using the point-slope form of an equation of a line, we have $y-1=-\frac{3}{2}(x-1)$, and the equation of the tangent line to the graph of the function f at $(1,1)$ is $y=-\frac{3}{2}x+\frac{5}{2}$.

34. $(x-y-1)^3=x$. Differentiating both sides of the given equation implicitly, we obtain $3(x-y-1)^2\left(1-y'\right)=1$. At the point $(1,-1)$, $3(1+1-1)^2\left(1-y'\right)=1$ or $y'=\frac{2}{3}$. Using the point-slope form of an equation of a line, we have $y+1=\frac{2}{3}(x-1)$ or $y=\frac{2}{3}x-\frac{5}{3}$.

35. $xy=1$. Differentiating implicitly, we have $xy'+y=0$, or $y'=-\dfrac{y}{x}$. Differentiating implicitly once again, we have $xy''+y'+y'=0$. Therefore, $y''=-\dfrac{2y'}{x}=\dfrac{2\left(\frac{y}{x}\right)}{x}=\dfrac{2y}{x^2}$.

36. $x^3+y^3=28$. Differentiating implicitly, we have $3x^2+3y^2y'=0$. Differentiating again, we have $6x+3y^2y''+6y\left(y'\right)^2=0$. Thus, $y''=-\dfrac{2y\left(y'\right)^2+2x}{y^2}$. But $\dfrac{dy}{dx}=-\dfrac{x^2}{y^2}$, and therefore,

$$y''=-\frac{2y\left(\frac{x^4}{y^4}\right)+2x}{y^2}=-\frac{2\left(\frac{x^4}{y^3}+x\right)}{y^2}=-\frac{2x\left(x^3+y^3\right)}{y^5}.$$

37. $y^2-xy=8$. Differentiating implicitly we have $2yy'-y-xy'=0$, and so $y'=\dfrac{y}{2y-x}$. Differentiating implicitly again, we have $2\left(y'\right)^2+2yy''-y'-y'-xy''=0$, so $y''=\dfrac{2y'-2\left(y'\right)^2}{2y-x}=\dfrac{2y'\left(1-y'\right)}{2y-x}$. Then

$$y''=\frac{2\left(\frac{y}{2y-x}\right)\left(1-\frac{y}{2y-x}\right)}{2y-x}=\frac{2y(2y-x-y)}{(2y-x)^3}=\frac{2y(y-x)}{(2y-x)^3}.$$

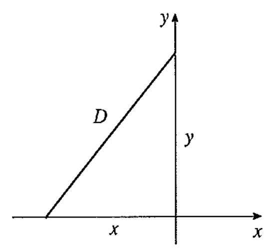
38. Differentiating $x^{1/3}+y^{1/3}=1$ implicitly, we have $\frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}y'=0$ and $y'=-\frac{y^{2/3}}{x^{2/3}}$. Differentiating implicitly once again, we have

$$y''=-\frac{x^{2/3}\left(\frac{2}{3}\right)y^{-1/3}y'-y^{2/3}\left(\frac{2}{3}\right)x^{-1/3}}{x^{4/3}}=\frac{-\frac{2}{3}x^{2/3}y^{-1/3}\left(-\frac{y^{2/3}}{x^{2/3}}\right)+\frac{2}{3}y^{2/3}x^{-1/3}}{x^{4/3}}=\frac{2}{3}\left(\frac{y^{1/3}+y^{2/3}x^{-1/3}}{x^{4/3}}\right)$$
$$=\frac{2y^{1/3}\left(x^{1/3}+y^{1/3}\right)}{3x^{4/3}x^{1/3}}=\frac{2y^{1/3}}{3x^{5/3}}.$$

39. a. Differentiating the given equation with respect to t, we obtain $\frac{dV}{dt}=\pi r^2\frac{dh}{dt}+2\pi rh\frac{dr}{dt}=\pi r\left(r\frac{dh}{dt}+2h\frac{dr}{dt}\right)$.

b. Substituting $r=2$, $h=6$, $\frac{dr}{dt}=0.1$, and $\frac{dh}{dt}=0.3$ into the expression for $\frac{dV}{dt}$, we obtain $\frac{dV}{dt}=\pi(2)[2(0.3)+2(6)(0.1)]=3.6\pi$, and so the volume is increasing at the rate of 3.6π in^3/sec.

40. Let $(x,0)$ and $(0,y)$ denote the position of the two cars. Then $D^2=x^2+y^2$. Differentiating with respect to t, we obtain $2D\frac{dD}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$, so $D\frac{dD}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$. When $t=4$, $x=-20$, and $y=28$, $\frac{dx}{dt}=-9$ and $\frac{dy}{dt}=11$. Therefore, $\left(\sqrt{(-20)^2+(28)^2}\right)\frac{dD}{dt}=(-20)(-9)+(28)(11)=488$, and so $\frac{dD}{dt}=\frac{488}{\sqrt{1184}}=14.18$ ft/sec. Thus, the distance is changing at the rate of 14.18 ft/sec.



41. We are given $\frac{dp}{dt}=2$ and wish to find $\frac{dx}{dt}$ when $x=9$ and $p=63$. Differentiating the equation $p+x^2=144$ with respect to t, we obtain $\frac{dp}{dt}+2x\frac{dx}{dt}=0$. When $x=9$, $p=63$, and $\frac{dp}{dt}=2$, we have $2+2(9)\frac{dx}{dt}=0$, and so and $\frac{dx}{dt}=-\frac{1}{9}\approx-0.111$. Thus, the quantity demanded is decreasing at the rate of approximately 111 tires per week.

42. $p=\frac{1}{2}x^2+48$. Differentiating implicitly, we have $\frac{dp}{dt}-x\frac{dx}{dt}=0$, so $-x\frac{dx}{dt}=-\frac{dp}{dt}$, and thus $\frac{dx}{dt}=\frac{dp/dt}{x}$. When $x=6$, $p=66$, and $\frac{dp}{dt}=-3$, we have $\frac{dx}{dt}=-\frac{3}{6}=-\frac{1}{2}$, or $\left(-\frac{1}{2}\right)(1000)=-500$ tires/week.

43. $100x^2+9p^2=3600$. Differentiating the given equation implicitly with respect to t, we have $200x\frac{dx}{dt}+18p\frac{dp}{dt}=0$. Next, when $p=14$, the given equation yields $100x^2+9(14)^2=3600$, so $100x^2=1836$, or $x\approx4.2849$. When $p=14$, $\frac{dp}{dt}=-0.15$, and $x\approx4.2849$, we have $200(4.2849)\frac{dx}{dt}+18(14)(-0.15)=0$, and so $\frac{dx}{dt}\approx0.0441$. Thus, the quantity demanded is increasing at the rate of approximately 44 headphones per week.

44. $625p^2-x^2=100$. Differentiating the given equation implicitly with respect to t, we have $1250p\frac{dp}{dt}-2x\frac{dx}{dt}=0$. To find p when $x=25$, we solve the equation $625p^2-625=100$, obtaining $p=\sqrt{\frac{725}{625}}\approx1.0770$. Therefore, $1250(1.077)(-0.02)-2(25)\frac{dx}{dt}=0$, and so $\frac{dx}{dt}=-0.5385$. We conclude that the supply is falling at the rate of 539 dozen eggs per week.

45. Differentiating $625p^2-x^2=100$ implicitly, we have $1250p\frac{dp}{dt}-2x\frac{dx}{dt}=0$. When $p=1.0770$, $x=25$, and $\frac{dx}{dt}=-1$, we find that $1250(1.077)\frac{dp}{dt}-2(25)(-1)=0$, and so $\frac{dp}{dt}=-\frac{50}{1250(1.077)}=-0.037$. We conclude that the price is decreasing at the rate of 3.7 cents per carton.

46. $p = -0.01x^2 - 0.1x + 6$. Differentiating the given equation with respect to p, we obtain $1 = -0.02x\frac{dx}{dp} - 0.1\frac{dx}{dp} = -(0.02x + 0.1)\frac{dx}{dp}$. When $x = 10$, we have $1 = -[0.02(10) + 0.1]\frac{dx}{dp}$, so $\frac{dx}{dp} = -\frac{1}{0.3} = -\frac{10}{3}$. Also, for this value of x, $p = -0.01(100) - 0.1(10) + 6 = 4$. Therefore, for these values of x and p, $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{p\frac{dx}{dp}}{f(p)} = -\dfrac{4\left(-\frac{10}{3}\right)}{10} = \dfrac{4}{3} > 1$, and so the demand is elastic.

47. $p = -0.01x^2 - 0.2x + 8$. Differentiating the given equation implicitly with respect to p, we have $1 = -0.02x\dfrac{dx}{dp} - 0.2\dfrac{dx}{dp} = -[0.02x + 0.2]\dfrac{dx}{dp}$, so $\dfrac{dx}{dp} = -\dfrac{1}{0.02x + 0.2}$. When $x = 15$, $p = -0.01(15)^2 - 0.2(15) + 8 = 2.75$, and so and $\dfrac{dx}{dp} = -\dfrac{1}{0.02(15) + 0.2} = -2$. Therefore, $E(p) = -\dfrac{pf'(p)}{f(p)} = -\dfrac{(2.75)(-2)}{15} \approx 0.37 < 1$, and the demand is inelastic.

48. a. The required output is $Q(16, 81) = 5\left(16^{1/4}\right)\left(81^{3/4}\right) = 270$, or \$270,000.

b. Rewriting $5x^{1/4}y^{3/4} = 270$ as $x^{1/4}y^{3/4} = 54$ and differentiating implicitly, we have $\frac{1}{4}x^{-3/4}y^{3/4} + x^{1/4}\left(\frac{3}{4}y^{-1/4}\dfrac{dy}{dx}\right) = 0$, so $\dfrac{dy}{dx} = -\frac{1}{4}x^{-3/4}y^{3/4}\left(\frac{4}{3}x^{-1/4}y^{1/4}\right) = -\dfrac{y}{3x}$. If $x = 16$ and $y = 81$, then $\dfrac{dy}{dx} = -\dfrac{81}{3 \cdot 16} = -1.6875$. Thus, to keep the output constant at \$270,000, the amount spent on capital should decrease by \$1687.50 per \$1000 in labor spending. The MRTS is \$1687.50 per thousand dollars.

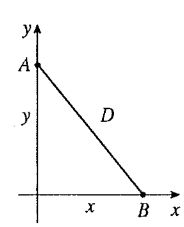
49. a. The required output is $Q(32, 243) = 20\left(32^{3/5}\right)\left(243^{2/5}\right) = 1440$, or \$1440 billion.

b. Differentiating $20x^{3/5}y^{2/5} = 1440$ implicitly with respect to x, we have $20\left(\frac{3}{5}x^{-2/5}y^{2/5}\right) + 20\left(x^{3/5}\right)\left(\frac{2}{5}y^{-3/5}\dfrac{dy}{dx}\right) = 0$, so $\dfrac{dy}{dx} = -\frac{3}{5}x^{-2/5}y^{2/5}\left(\frac{5}{2}x^{-3/5}y^{3/5}\right) = -\dfrac{3y}{2x}$. If $x = 32$ and $y = 243$, then $\dfrac{dy}{dx} = -\dfrac{3 \cdot 243}{2 \cdot 32} \approx -11.39$, so the amount spent on capital should decrease by approximately \$11.4 billion. The MRTS is \$11.4 billion per billion dollars.

50. $V = x^3$, so $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$. When $x = 5$ and $\frac{dx}{dt} = 0.1$, we have $\frac{dV}{dt} = 3(25)(0.1) = 7.5\ \text{in}^3/\text{sec}$.

51. $A = \pi r^2$. Differentiating with respect to t, we obtain $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. When the radius of the circle is 60 ft and increasing at the rate of $\frac{1}{2}$ ft/sec, $\frac{dA}{dt} = 2\pi(60)\left(\frac{1}{2}\right) = 60\pi\ \text{ft}^2/\text{sec}$. Thus, the area is increasing at a rate of approximately $188.5\ \text{ft}^2/\text{sec}$.

52. Let D denote the distance between the two ships, x the distance that ship A traveled north, and y the distance that ship B traveled east. Then $D^2 = x^2 + y^2$. Differentiating implicitly, we have $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, so $D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$. At 1 p.m., $x = 12$ and $y = 15$, so $\sqrt{144 + 225}\frac{dD}{dt} = (12)(12) + (15)(15)$. Thus, $\frac{dD}{dt} = \frac{369}{\sqrt{369}} \approx 19.21$ ft/sec.

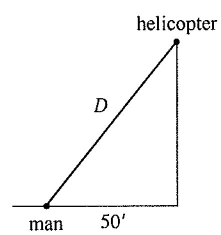


53. $A = \pi r^2$, so $r = \left(\frac{A}{\pi}\right)^{1/2}$. Differentiating with respect to t, we obtain $\frac{dr}{dt} = \frac{1}{2}\left(\frac{A}{\pi}\right)^{-1/2}\frac{dA}{dt}$. When the area of the spill is 1600π ft^2 and increasing at the rate of 80π ft^2/sec, $\frac{dr}{dt} = \frac{1}{2}\left(\frac{1600\pi}{\pi}\right)^{-1/2}(80\pi) = \pi$ ft/sec. Thus, the radius is increasing at the rate of approximately 3.14 ft/sec.

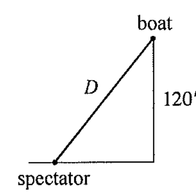
54. Let D denote the distance between the two cars, x the distance traveled by the car heading east, and y the distance traveled by the car heading north. Then $D^2 = x^2 + y^2$. Differentiating with respect to t, we have $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, so $D\frac{dD}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$. Notice also that $\frac{dx}{dt} = 2t + 1$ and $\frac{dy}{dt} = 2t + 3$. When $t = 5$, $x = 5^2 + 5 = 30$, $y = 5^2 + 3(5) = 40$, $\frac{dx}{dt} = 2(5) + 1 = 11$, and $\frac{dy}{dt} = 2(5) + 3 = 13$, so $\frac{dD}{dt} = \frac{(30)(11) + (40)(13)}{\sqrt{900 + 1600}} = 17$ ft/sec.

55. Let $(x, 0)$ and $(0, y)$ denote the position of the two cars at time t. Then $y = t^2 + 2t$. Now $D^2 = x^2 + y^2$ so $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ and thus $D\frac{dD}{dt} = x\frac{dx}{dt} + (t^2 + 2t)(2t + 2)$. When $t = 4$, we have $x = -20$, $\frac{dx}{dt} = -9$, and $y = 24$, so $\sqrt{(-20)^2 + (24)^2}\frac{dD}{dt} = (-20)(-9) + (24)(10)$, and therefore $\frac{dD}{dt} = \frac{420}{\sqrt{976}} \approx 13.44$. That is, the distance is changing at approximately 13.44 ft/sec.

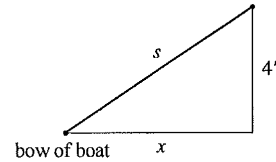
56. $D^2 = x^2 + (50)^2 = x^2 + 2500$. Differentiating implicitly with respect to t, we have $2D\frac{dD}{dt} = 2x\frac{dx}{dt}$, so $\frac{dD}{dt} = \frac{x\frac{dx}{dt}}{D}$. When $x = 120$ and $\frac{dx}{dt} = 44$, $\frac{dD}{dt} = \frac{(120)(44)}{\sqrt{(120)^2 + (50)^2}} \approx 40.6$, and so the distance between the helicopter and the man is increasing at the rate of 40.6 ft/sec.



57. Referring to the diagram, we see that $D^2 = 120^2 + x^2$. Differentiating this last equation with respect to t, we have $2D\frac{dD}{dt} = 2x\frac{dx}{dt}$, and so $\frac{dD}{dt} = \frac{x\frac{dx}{dt}}{D}$. When $x = 50$ and $\frac{dx}{dt} = 20$, $D = \sqrt{120^2 + 50^2} = 130$ and $\frac{dD}{dt} = \frac{(20)(50)}{130} \approx 7.69$, or 7.69 ft/sec.



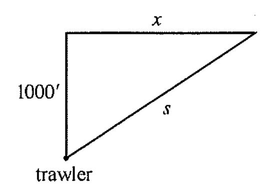
58. By the Pythagorean Theorem, $s^2 = x^2 + 4^2 = x^2 + 16$. We want to find $\frac{dx}{dt}$ when $x = 25$, given that $\frac{ds}{dt} = -3$. Differentiating both sides of the equation with respect to t yields $2s\frac{ds}{dt} = 2x\frac{dx}{dt}$, or $\frac{dx}{dt} = \frac{s\frac{ds}{dt}}{x}$. Now when $x = 25$, $s^2 = 25^2 + 16 = 641$ and $s = \sqrt{641}$. Therefore, when $x = 25$, we have $\frac{dx}{dt} = \frac{\sqrt{641}(-3)}{25} \approx -3.04$; that is, the boat is approaching the dock at the rate of approximately 3.04 ft/sec.



59. Let V and S denote its volume and surface area. Then we are given that $\frac{dV}{dt} = kS$, where k is the constant of proportionality. But from $V = \frac{4}{3}\pi r^3$, we find, upon differentiating both sides with respect to t, that $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \frac{dr}{dt} = kS = k\left(4\pi r^2\right)$. Therefore, $\frac{dr}{dt} = k$ a constant.

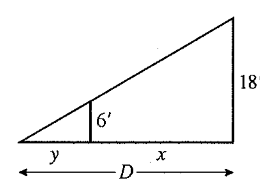
60. Let V denote the volume of the soap bubble and r its radius. Then, we are given $\frac{dV}{dt} = 8$. Differentiating the formula $V = \frac{4}{3}\pi r^3$ with respect to t, we find $\frac{dV}{dt} = \left(\frac{4}{3}\right)\left(3\pi r^2 \frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$. When $r = 10$, we have $\dfrac{dr}{dt} = \dfrac{8}{4\pi\left(10^2\right)} \approx 0.0064$. Thus, the radius is increasing at the rate of approximately 0.0064 cm/sec. From $s = 4\pi r^2$, we find $\frac{ds}{dt} = 4\pi(2r)\frac{dr}{dt} = 8\pi r \frac{dr}{dt}$. Therefore, when $r = 10$, we have $\frac{ds}{dt} = 8\pi(10)(0.0064) \approx 1.6$. Thus, the surface area is increasing at the rate of approximately 1.6 cm^2/sec.

61. We are given that $\frac{dx}{dt} = 264$. Using the Pythagorean Theorem, $s^2 = x^2 + 1000^2 = x^2 + 1{,}000{,}000$. We want to find $\dfrac{ds}{dt}$ when $s = 1500$. Differentiating both sides of the equation with respect to t, we have $2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt}$ and so $\dfrac{ds}{dt} = \dfrac{x\frac{dx}{dt}}{s}$. When $s = 1500$, we have $1500^2 = x^2 + 1{,}000{,}000$, or $x = \sqrt{1{,}250{,}000}$. Therefore, $\dfrac{ds}{dt} = \dfrac{\sqrt{1{,}250{,}000}\cdot(264)}{1500} \approx 196.8$, that is, the aircraft is receding from the trawler at the speed of approximately 196.8 ft/sec.



62. The volume V of the water in the pot is $V = \pi r^2 h = \pi(16)h = 16\pi h$. Differentiating with respect to t, we obtain $\frac{dV}{dt} = 16\pi\frac{dh}{dt}$. Therefore, with $\frac{dh}{dt} = 0.4$, we find $\frac{dV}{dt} = 16\pi(0.4) \approx 20.1$; that is, water is being poured into the pot at the rate of approximately 20.1 cm^3/sec.

63. $\dfrac{y}{6} = \dfrac{y+x}{18}$, $18y = 6(y+x)$, so $3y = y + x$, $2y = x$, and $y = \frac{1}{2}x$. Thus, $D = y + x = \frac{3}{2}x$. Differentiating implicitly, we have $\frac{dD}{dt} = \frac{3}{2}\cdot\frac{dx}{dt}$, and when $\frac{dx}{dt} = 6$, $\frac{dD}{dt} = \frac{3}{2}(6) = 9$, or 9 ft/sec.



64. Differentiating $x^2 + y^2 = 400$ with respect to t gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. When $x = 12$, we have $144 + y^2 = 400$, or $y = \sqrt{256} = 16$. Therefore, with $x = 12$, $\frac{dx}{dt} = 5$, and $y = 16$, we find $2(12)(5) + 2(16)\frac{dy}{dt} = 0$, or $\frac{dy}{dt} = -3.75$. Thus, the top of the ladder is sliding down the wall at the rate of 3.75 ft/sec.

65. Differentiating $x^2 + y^2 = 13^2 = 169$ with respect to t gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. When $x = 12$, we have $144 + y^2 = 169$, or $y = 5$. Therefore, with $x = 12$, $y = 5$, and $\frac{dx}{dt} = 8$, we find $2(12)(8) + 2(5)\frac{dy}{dt} = 0$, or $\frac{dy}{dt} = -19.2$. Thus, the top of the ladder is sliding down the wall at the rate of 19.2 ft/sec.