39. We first solve the demand equation for x in terms of p. Thus, $p = \sqrt{9 - 0.02x}$, and $p^2 = 9 - 0.02x$, or $x = -50p^2 + 450$. With $f(p) = -50p^2 + 450$, we find $E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p(-100p)}{-50p^2 + 450} = \frac{2p^2}{9 - p^2}$. Setting $E(p) = 1$ gives $2p^2 = 9 - p^2$, so $p = \sqrt{3}$. So the demand is inelastic in $\left(0, \sqrt{3}\right)$, unitary when $p = \sqrt{3}$, and elastic in $\left(\sqrt{3}, 3\right)$.

40. $f(p) = 10\left(\frac{50 - p}{p}\right)^{1/2} = 10\left(\frac{50}{p} - 1\right)^{1/2}$, so

$f'(p) = 10\left(\frac{1}{2}\right)\left(\frac{50}{p} - 1\right)^{-1/2}\left(-\frac{50}{p^2}\right) = -\frac{250}{p^2}\left(\frac{50}{p} - 1\right)^{-1/2}$. Then the elasticity of demand is given by

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p\left(-\frac{250}{p^2}\right)\left(\frac{50}{p} - 1\right)^{-1/2}}{10\left(\frac{50}{p} - 1\right)^{1/2}} = -\frac{\frac{250}{p}}{10\left(\frac{50}{p} - 1\right)} = \frac{25}{p\left(\frac{50 - p}{p}\right)} = \frac{25}{50 - p}.$$

Setting $E = 1$ gives $1 = \frac{25}{50 - p}$, and so $25 = 50 - p$, and $p = 25$. Thus, if $p > 25$, then $E > 1$, and the demand is elastic; if $p = 25$, then $E = 1$ and the demand is unitary; and if $p < 25$, then $E < 1$ and the demand is inelastic.

41. True. $\overline{C}'(x) = \frac{d}{dx}\left[\frac{C(x)}{x}\right] = \frac{xC'(x) - C(x)\frac{d}{dx}(x)}{x^2} = \frac{xC'(x) - C(x)}{x^2}$.

42. False. In fact, it makes good sense to *increase* the level of production since, in this instance, the profit increases by $f'(a)$ units per unit increase in x.

3.5 Higher-Order Derivatives

Concept Questions page 217

1. a. The second derivative of f is the derivative of f'.

b. To find the second derivative of f, we differentiate f'.

2. $f'(t)$ measures its velocity at time t, and $f''(t)$ measures its acceleration at time t.

3. a. $f'(t) > 0$ and $f''(t) > 0$ in (a, b).

b. $f'(t) > 0$ and $f''(t) < 0$ in (a, b).

c. $f'(t) < 0$ and $f''(t) < 0$ in (a, b).

d. $f'(t) < 0$ and $f''(t) > 0$ in (a, b).

4. a. $f'(t) > 0$ and $f''(t) = 0$ in (a, b).

b. $f'(t) < 0$ and $f''(t) = 0$ in (a, b).

c. $f'(t) = 0$ and $f''(t) = 0$ in (a, b).

Exercises page 218

1. $f(x) = 4x^2 - 2x + 1$, so $f'(x) = 8x - 2$ and $f''(x) = 8$.

2. $f(x) = -0.2x^2 + 0.3x + 4$, so $f'(x) = -0.4x + 0.3$ and $f''(x) = -0.4$.

3. $f(x) = 2x^3 - 3x^2 + 1$, so $f'(x) = 6x^2 - 6x$ and $f''(x) = 12x - 6 = 6(2x - 1)$.

4. $g(x) = -3x^3 + 24x^2 + 6x - 64$, so $g'(x) = -9x^2 + 48x + 6$ and $g''(x) = -18x + 48$.

5. $h(t) = t^4 - 2t^3 + 6t^2 - 3t + 10$, so $h'(t) = 4t^3 - 6t^2 + 12t - 3$ and $h''(t) = 12t^2 - 12t + 12 = 12(t^2 - t + 1)$.

6. $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$, so $f'(x) = 5x^4 - 4x^3 + 3x^2 - 2x + 1$ and $f''(x) = 20x^3 - 12x^2 + 6x - 2$.

7. $f(x) = (x^2+2)^5$, so $f'(x) = 5(x^2+2)^4(2x) = 10x(x^2+2)^4$ and
$f''(x) = 10(x^2+2)^4 + 10x(4)(x^2+2)^3(2x) = 10(x^2+2)^3\left[(x^2+2) + 8x^2\right] = 10(9x^2+2)(x^2+2)^3$.

8. $g(t) = t^2(3t+1)^4$, so
$g'(t) = 2t(3t+1)^4 + t^2(4)(3t+1)^3(3) = 2t(3t+1)^3[(3t+1) + 6t] = (3t+1)^3(18t^2 + 2t)$ and
$g''(t) = 2t(9t+1)(3)(3t+1)^2(3) + (3t+1)^3(36t+2) = 2(3t+1)^2[9t(9t+1) + (3t+1)(18t+1)]$
$= 2(3t+1)^2(81t^2 + 9t + 54t^2 + 3t + 18t + 1) = 2(135t^2 + 30t + 1)(3t+1)^2$.

9. $g(t) = (2t^2-1)^2(3t^2)$, so
$g'(t) = 2(2t^2-1)(4t)(3t^2) - (2t^2-1)^2(6t) = 6t(2t^2-1)\left[4t^2 + (2t^2-1)\right] = 6t(2t^2-1)(6t^2-1)$
$= 6t(12t^4 - 8t^2 + 1) = 72t^5 - 48t^3 + 6t$
and $g''(t) = 360t^4 - 144t^2 + 6 = 6(60t^4 - 24t^2 + 1)$.

10. $h(x) = (x^2+1)^2(x-1)$, so $h'(x) = 2(x^2+1)(2x)(x-1) +$
$(x^2+1)^2(1) = (x^2+1)\left[4x(x-1) + (x^2+1)\right] = (x^2+1)(5x^2 - 4x + 1)$ and
$h''(x) = 2x(5x^2 - 4x + 1) + (x^2+1)(10x - 4) = 10x^3 - 8x^2 + 2x + 10x^3 - 4x^2 + 10x - 4$
$= 20x^3 - 12x^2 + 12x - 4 = 4(5x^3 - 3x^2 + 3x - 1)$.

11. $f(x) = (2x^2+2)^{7/2}$, so $f'(x) = \frac{7}{2}(2x^2+2)^{5/2}(4x) = 14x(2x^2+2)^{5/2}$ and
$f''(x) = 14(2x^2+2)^{5/2} + 14x\left(\frac{5}{2}\right)(2x^2+2)^{3/2}(4x) = 14(2x^2+2)^{3/2}\left[(2x^2+2) + 10x^2\right]$
$= 28(6x^2+1)(2x^2+2)^{3/2}$.

12. $h(w) = (w^2+2w+4)^{5/2}$, so $h'(w) = \frac{5}{2}(w^2+2w+4)^{3/2}(2w+2) = 5(w+1)(w^2+2w+4)^{3/2}$ and
$h''(w) = 5(w^2+2w+4)^{3/2} + 5(w+1)\left(\frac{3}{2}\right)(w^2+2w+4)^{1/2}(2w+2)$
$= 5(w^2+2w+4)^{1/2}\left[(w^2+2w+4) + 3(w+1)^2\right] = 5(4w^2+8w+7)(w^2+2w+4)^{1/2}$.

13. $f(x) = x(x^2+1)^2$, so
$f'(x) = (x^2+1)^2 + x(2)(x^2+1)(2x) = (x^2+1)\left[(x^2+1) + 4x^2\right] = (x^2+1)(5x^2+1)$ and
$f''(x) = 2x(5x^2+1) + (x^2+1)(10x) = 2x(5x^2 + 1 + 5x^2 + 5) = 4x(5x^2+3)$.

14. $g(u) = u(2u-1)^3$, so $g'(u) = (2u-1)^3 + u(3)(2u-1)^2(2) = (2u-1)^2[(2u-1) + 6u] = (8u-1)(2u-1)^2$
and $g''(u) = 8(2u-1)^2 + (8u-1)(2)(2u-1)(2) = 4(2u-1)[2(2u-1) + (8u-1)] = 12(2u-1)(4u-1)$.

15. $f(x) = \dfrac{x}{2x+1}$, so $f'(x) = \dfrac{(2x+1)(1) - x(2)}{(2x+1)^2} = \dfrac{1}{(2x+1)^2}$ and

$f''(x) = \dfrac{d}{dx}(2x+1)^{-2} = -2(2x+1)^{-3}(2) = -\dfrac{4}{(2x+1)^3}$.

16. $g(t) = \dfrac{t^2}{t-1}$, so $g'(t) = \dfrac{(t-1)(2t) - t^2(1)}{(t-1)^2} = \dfrac{t^2-2t}{(t-1)^2} = \dfrac{t(t-2)}{(t-1)^2}$ and

$g''(t) = \dfrac{(t-1)^2(2t-2) - t(t-2)2(t-1)}{(t-1)^4} = \dfrac{2(t-1)\left[(t-1)^2 - t(t-2)\right]}{(t-1)^4} = \dfrac{2}{(t-1)^3}$.

17. $f(s) = \dfrac{s-1}{s+1}$, so $f'(s) = \dfrac{(s+1)(1) - (s-1)(1)}{(s+1)^2} = \dfrac{2}{(s+1)^2}$ and

$f''(s) = 2\dfrac{d}{ds}(s+1)^{-2} = -4(s+1)^{-3} = -\dfrac{4}{(s+1)^3}$.

18. $f(u) = \dfrac{u}{u^2+1}$, so $f'(u) = \dfrac{(u^2+1)(1) - (u)(2u)}{(u^2+1)^2} = \dfrac{-u^2+1}{(u^2+1)^2}$ and

$f''(u) = \dfrac{(u^2+1)^2(-2u) - (-u^2+1)(2)(u^2+1)(2u)}{(u^2+1)^4} = \dfrac{2u(u^2+1)(-u^2-1+2u^2-2)}{(u^2+1)^4} = \dfrac{2u(u^2-3)}{(u^2+1)^3}$.

19. $f(u) = \sqrt{4-3u} = (4-3u)^{1/2}$, so $f'(u) = \frac{1}{2}(4-3u)^{-1/2}(-3) = -\dfrac{3}{2\sqrt{4-3u}}$ and

$f''(u) = -\dfrac{3}{2}\cdot\dfrac{d}{du}(4-3u)^{-1/2} = -\dfrac{3}{2}\left(-\dfrac{1}{2}\right)(4-3u)^{-3/2}(-3) = -\dfrac{9}{4(4-3u)^{3/2}}$.

20. $f(x) = \sqrt{2x-1} = (2x-1)^{1/2}$, so $f'(x) = \frac{1}{2}(2x-1)^{-1/2}(2) = (2x-1)^{-1/2} = \dfrac{1}{\sqrt{2x-1}}$ and

$f''(x) = -\frac{1}{2}(2x-1)^{-3/2}(2) = -(2x-1)^{-3/2} = -\dfrac{1}{\sqrt{(2x-1)^3}}$.

21. $f(x) = 3x^4 - 4x^3$, so $f'(x) = 12x^3 - 12x^2$, $f''(x) = 36x^2 - 24x$, and $f'''(x) = 72x - 24$.

22. $f(x) = 3x^5 - 6x^4 + 2x^2 - 8x + 12$, so $f'(x) = 15x^4 - 24x^3 + 4x - 8$, $f''(x) = 60x^3 - 72x^2 + 4$, and $f'''(x) = 180x^2 - 144x$.

23. $f(x) = \dfrac{1}{x}$, so $f'(x) = \dfrac{d}{dx}(x^{-1}) = -x^{-2} = -\dfrac{1}{x^2}$, $f''(x) = 2x^{-3} = \dfrac{2}{x^3}$, and $f'''(x) = -6x^{-4} = -\dfrac{6}{x^4}$.

24. $f(x) = \dfrac{2}{x^2}$, so $f'(x) = 2\dfrac{d}{dx}(x^{-2}) = -4x^{-3} = -\dfrac{4}{x^3}$, $f''(x) = 12x^{-4} = \dfrac{12}{x^4}$, and $f'''(x) = -48x^{-5} = -\dfrac{48}{x^5}$.

25. $g(s) = (3s-2)^{1/2}$, so $g'(s) = \frac{1}{2}(3s-2)^{-1/2}(3) = \dfrac{3}{2(3s-2)^{1/2}}$,

$g''(s) = \frac{3}{2}\left(-\frac{1}{2}\right)(3s-2)^{-3/2}(3) = -\frac{9}{4}(3s-2)^{-3/2} = -\dfrac{9}{4(3s-2)^{3/2}}$, and

$g'''(s) = \frac{27}{8}(3s-2)^{-5/2}(3) = \frac{81}{8}(3s-2)^{-5/2} = \dfrac{81}{8(3s-2)^{5/2}}$.

26. $g(t)=\sqrt{2t+3}$, so $g'(t)=\frac{1}{2}(2t+3)^{-1/2}(2)=(2t+3)^{-1/2}$, $g''(t)=-\frac{1}{2}(2t+3)^{-3/2}(2)=-(2t+3)^{-3/2}$, and $g'''(t)=\frac{3}{2}(2t+3)^{-5/2}(2)=\dfrac{3}{(2t+3)^{5/2}}$.

27. $f(x)=(2x-3)^4$, so $f'(x)=4(2x-3)^3(2)=8(2x-3)^3$, $f''(x)=24(2x-3)^2(2)=48(2x-3)^2$, and $f'''(x)=96(2x-3)(2)=192(2x-3)$.

28. $g(t)=\left(\frac{1}{2}t^2-1\right)^5$, so $g'(t)=5\left(\frac{1}{2}t^2-1\right)^4(t)=5t\left(\frac{1}{2}t^2-1\right)^4$,

$g''(t)=5\left(\frac{1}{2}t^2-1\right)^4+5t(4)\left(\frac{1}{2}t^2-1\right)^3(t)=5\left(\frac{1}{2}t^2-1\right)^3\left[\left(\frac{1}{2}t^2-1\right)+4t^2\right]=\frac{5}{2}(9t^2-2)\left(\frac{1}{2}t^2-1\right)^3$, and

$$g'''(t)=\frac{5}{2}\left[18t\left(\frac{1}{2}t^2-1\right)^3+(9t^2-2)\,3\left(\frac{1}{2}t^2-1\right)^2(t)\right]=\frac{15}{2}t\left(\frac{1}{2}t^2-1\right)^2\left[6\left(\frac{1}{2}t^2-1\right)+(9t^2-2)\right]$$
$$=30t(3t^2-2)\left(\frac{1}{2}t^2-1\right)^2.$$

29. Its velocity at any time t is $v(t)=\frac{d}{dt}(16t^2)=32t$. The hammer strikes the ground when $16t^2=256$ or $t=4$ (we reject the negative root). Therefore, its velocity at the instant it strikes the ground is $v(4)=32(4)=128$ ft/sec. Its acceleration at time t is $a(t)=\frac{d}{dt}(32t)=32$. In particular, its acceleration at $t=4$ is 32 ft/sec^2.

30. $s(t)=20t+8t^2-t^3$, so $s'(t)=20+16t-3t^2$ and $s''(t)=16-6t$. In particular, $s''\left(\frac{8}{3}\right)=16-6\left(\frac{8}{3}\right)=16-\frac{48}{3}=0$. We conclude that the acceleration of the car at $t=\frac{8}{3}$ seconds is zero and that the car will start to decelerate at that point in time.

31. $P(t)=0.38t^2+1.3t+3$.

a. The projected percentage is $P(5)=0.38(5)^2+1.3(5)+3=19$, or 19%.

b. $P'(t)=0.76t+1.3$, so the percentage of vehicles with transmissions that have 7 or more speeds is projected to be changing at the rate of $P'(5)=0.76(5)+1.3=5.1$, or 5.1% per year (in 2015).

c. $P''(15)=0.76$, so the rate of increase in vehicles with such transmissions is itself increasing at the rate of 0.76% per year per year in 2025.

32. $N(t)=0.00525t^2+0.075t+4.7$.

a. The projected number of people of age 65 and over with Alzheimer's disease in the U.S. is projected to be $N(2)=0.00525(2)^2+0.075(2)+4.7=4.871$, or 4.871 million, in 2030.

b. $N'(t)=0.0105t+0.075$, so the number of patients is projected to be growing at the rate of $N'(2)=0.0105(2)+0.075=0.096$, or 96,000 per decade, in 2030.

c. $N''(t)=0.0105$, so the rate of growth is projected to be growing at the rate of 10,500 per decade per decade.

33. $N(t)=-0.1t^3+1.5t^2+100$.

a. $N'(t)=-0.3t^2+3t=0.3t(10-t)$. Because $N'(t)>0$ for $t=0,1,2,\ldots,8$, it is evident that $N(t)$ (and therefore the crime rate) was increasing from 2006 through 2014.

b. $N''(t)=-0.6t+3=0.6(5-t)$. Now $N''(4)=0.6>0$, $N''(5)=0$, $N''(6)=-0.6<0$, $N''(7)=-1.2<0$, and $N''(8)=-1.8<0$. This shows that the rate of the rate of change was decreasing beyond $t=5$ (in the year 2011). This indicates that the program was working.

34. $G(t)=-0.2t^3+2.4t^2+60$.

a. $G'(t) = -0.6t^2 + 4.8t = 0.6t(8-t)$, so $G'(1) = 4.2$, $G'(2) = 7.2$, $G'(3) = 9$, $G'(4) = 9.6$, $G'(5) = 9$, $G'(6) = 7.2$, $G'(7) = 4.2$, and $G'(8) = 0$.

b. $G''(t) = -1.2t + 4.8 = 1.2(4-t)$, so $G''(1) = 3.6$, $G''(2) = 2.4$, $G''(3) = 1.2$, $G''(4) = 0$, $G''(5) = -1.2$, $G''(6) = -2.4$, $G''(7) = -3.6$, and $G''(8) = -4.8$.

c. Our calculations show that the GDP is increasing at an increasing rate in the first five years. Even though the GDP continues to rise from that point on, the negativity of $G''(t)$ shows that the rate of increase is slowing down.

35. $S(t) = 4t^3 + 2t^2 + 300t$, so $S(6) = 4(6)^3 + 2(6)^2 + 300(6) = 2736$. This says that 6 months after the grand opening of the store, monthly LP sales are projected to be 2736 units.
$S'(t) = 12t^2 + 4t + 300$, so $S'(6) = 12(6)^2 + 4(6) + 300 = 756$. Thus, monthly sales are projected to be increasing by 756 units per month.
$S''(t) = 24t + 4$, so $S''(6) = 24(6) + 4 = 148$. This says that the rate of increase of monthly sales is itself increasing at the rate of 148 units per month per month.

36. a. $f(t) = -0.2176t^3 + 1.962t^2 - 2.833t + 29.4$, so the median age of the population in the year 2000 was $f(4) = -0.2176(4)^3 + 1.962(4)^2 - 2.833(4) + 29.4 \approx 35.53$, or approximately 35.5 years old.

b. $f'(t) = -0.6528t^2 + 3.924t - 2.833$, so the median age of the population in the year 2000 was changing at the rate of $-0.6528(4)^2 + 3.924(4) - 2.833 = 2.4182$; that is, it was increasing at the rate of approximately 2.4 years of age per decade.

c. $f''(t) = -1.3056t + 3.924$, so the rate at which the median age of the population was changing in the year was $f''(4) = -1.3056(4) + 3.924 = -1.2984$. That is, the rate of change was decreasing at the rate of approximately 1.3 years of age per decade per decade.

37. $P(t) = 0.0004t^3 + 0.0036t^2 + 0.8t + 12$, so $P'(t) = 0.00012t^2 + 0.0072t + 0.8$. Thus, $P'(t) \geq 0.8$ for $0 \leq t \leq 13$. $P''(t) = 0.00024t + 0.0072$, and for $0 \leq t \leq 13$, $P''(t) > 0$. This means that the proportion of the U.S. population that was obese was increasing at an increasing rate from 1991 through 2004.

38. a. $h(t) = \frac{1}{16}t^4 - t^3 + 4t^2$, so $h'(t) = \frac{1}{4}t^3 - 3t^2 + 8t$.

b. $h'(0) = 0$, or 0 ft/sec. $h'(4) = \frac{1}{4}(64) - 3(16) + 8(4) = 0$, or 0 ft/sec, and $h'(8) = \frac{1}{4}(8)^3 - 3(64) + 8(8) = 0$, or 0 ft/sec.

c. $h''(t) = \frac{3}{4}t^2 - 6t + 8$.

d. $h''(0) = 8\text{ ft/sec}^2$, $h''(4) = \frac{3}{4}(16) - 6(4) + 8 = -4\text{ ft/sec}^2$, and $h''(8) = \frac{3}{4}(64) - 6(8) + 8 = 8\text{ ft/sec}^2$.

e. $h(0) = 0$ ft, $h(4) = \frac{1}{16}(4)^4 - (4)^3 + 4(4)^2 = 16$ ft, and $h(8) = \frac{1}{16}(8)^4 - (8)^3 + 4(8)^2 = 0$ ft.

39. $A(t) = -0.00006t^5 + 0.00468t^4 - 0.1316t^3 + 1.915t^2 - 17.63t + 100$, so $A'(t) = -0.0003t^4 + 0.01872t^3 - 0.3948t^2 + 3.83t - 17.63$ and $A''(t) = -0.0012t^3 + 0.05616t^2 - 0.7896t + 3.83$. Thus, $A'(10) = -3.09$ and $A''(10) = 0.35$. Our calcutations show that 10 minutes after the start of the test, the smoke remaining is decreasing at a rate of 3.09% per minute, but the rate at which the rate of smoke is decreasing is decreasing at the rate of 0.35 percent per minute per minute.

40. $P(t) = 33.55(t+5)^{0.205}$, so $P'(t) = 33.55(0.205)(t+5)^{-0.795} = 6.87775(t+5)^{-0.795}$ and $P''(t) = 6.87775(-0.795)(t+5)^{-1.795} = -5.46781125(t+5)^{-1.795}$. Thus, $P''(20) = -5.46781125(20+5)^{-1.795} \approx -0.017$, which says that the rate of the rate of change of such mothers is decreasing at the rate of $0.02\%/\text{yr}^2$.