3.4 Marginal Functions in Economics

Concept Questions

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1. **a.** The marginal cost function is the derivative of the cost function.
   b. The average cost function is equal to the total cost function divided by the total number of the commodity produced.
   c. The marginal average cost function is the derivative of the average cost function.
   d. The marginal revenue function is the derivative of the revenue function.
   e. The marginal profit function is the derivative of the profit function.

2. **a.** The elasticity of demand at a price P is $E(p) = -\frac{pf'(p)}{f(p)}$, where f is the demand function $x = f(p)$.
   b. The elasticity of demand is elastic if $E(p) > 1$, unitary if $E(p) = 1$, and inelastic if $E(p) < 1$. If $E(p) > 1$, then an increase in the unit price will cause the revenue to decrease, whereas a decrease in the unit price will cause the revenue to increase. If $E(p) = 1$, then an increase in the unit price will have no effect on the revenue. If $E(p) < 1$, then an increase in the unit price will cause the revenue to increase, and a decrease in the unit price will cause the revenue to decrease.

3. $P(x) = R(x) - C(x)$, so $P'(x) = R'(x) - C'(x)$. Using the given information, we find $P'(500) = R'(500) - C'(500) = 3 - 2.8 = 0.2$. Thus, if the level of production is 500, then the marginal profit is \$0.20 per unit. This tells us that the proprietor should increase production in order to increase the company's profit.

Exercises

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1. **a.** $C(x)$ is always increasing because as x, the number of units produced, increases, the amount of money that must be spent on production also increases.
   b. This occurs at $x = 4$, a production level of 4000. You can see this by looking at the slopes of the tangent lines for x less than, equal to, and a little larger then $x = 4$.

2. **a.** If very few units of the commodity are produced then the cost per unit of production will be very large. If x is very large, the typical total cost is very large due to overtime, excessive cost of raw material, machinery breakdown, etc., so that $A(x)$ is very large as well; in fact, for a typical total cost function $C(x)$, ultimately $A(x)$ grows faster than x, that is, $\lim_{x\to\infty} \frac{C(x)}{x} = \infty$.
   b. The average cost per unit is smallest (\$$y_0$) when the level of production is x_0 units.

3. **a.** The actual cost incurred in the production of the 1001st disc is given by
   $$C(1001) - C(1000) = \left[2000 + 2(1001) - 0.0001(1001)^2\right] - \left[2000 + 2(1000) - 0.0001(1000)^2\right]$$
   $$= 3901.7999 - 3900 = 1.7999, \text{ or approximately } \$1.80.$$
   The actual cost incurred in the production of the 2001st disc is given by
   $$C(2001) - C(2000) = \left[2000 + 2(2001) - 0.0001(2001)^2\right] - \left[2000 + 2(2000) - 0.0001(2000)^2\right]$$
   $$= 5601.5999 - 5600 = 1.5999, \text{ or approximately } \$1.60.$$
   b. The marginal cost is $C'(x) = 2 - 0.0002x$. In particular, $C'(1000) = 2 - 0.0002(1000) = 1.80$ and $C'(2000) = 2 - 0.0002(2000) = 1.60$.

4. a. $C(101) - C(100) = \left[0.0002(101)^3 - 0.06(101)^2 + 120(101) + 5000\right] - \left[0.0002(100)^3 - 0.06(100)^2 + 120(100) + 5000\right] \approx 114$, or approximately \$114.

Similarly, we find $C(201) - C(200) \approx \$120.06$ and $C(301) - C(300) \approx \$138.12$.

b. We compute $C'(x) = 0.0006x^2 - 0.12x + 120$. Thus, the required quantities are $C'(100) = 0.0006(100)^2 - 0.12(100) + 120 = 114$, or \$114; $C'(200) = 0.0006(200)^2 - 0.12(200) + 120 = 120$, or \$120; and $C'(300) = 0.0006(300)^2 - 0.12(300) + 120 = 138$, or \$138.

5. a. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{100x + 200{,}000}{x} = 100 + \dfrac{200{,}000}{x}$.

b. $\overline{C}'(x) = \dfrac{d}{dx}(100) + \dfrac{d}{dx}\left(200{,}000x^{-1}\right) = -200{,}000x^{-2} = -\dfrac{200{,}000}{x^2}$.

c. $\lim\limits_{x\to\infty} \overline{C}(x) = \lim\limits_{x\to\infty} \left(100 + \dfrac{200{,}000}{x}\right) = 100$. This says that the average cost approaches \$100 per unit if the production level is very high.

6. a. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{5000}{x} + 2$.

b. $\overline{C}'(x) = -\dfrac{5000}{x^2}$.

c. Because the marginal average cost function is negative for $x > 0$, the rate of change of the average cost function is negative for all $x > 0$.

7. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{2000 + 2x - 0.0001x^2}{x} = \dfrac{2000}{x} + 2 - 0.0001x$, so $\overline{C}'(x) = -\dfrac{2000}{x^2} + 0 - 0.0001 = -\dfrac{2000}{x^2} - 0.0001$.

8. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{0.0002x^3 - 0.06x^2 + 120x + 5000}{x} = 0.0002x^2 - 0.06x + 120 + \dfrac{5000}{x}$, so $\overline{C}'(x) = 0.0004x - 0.06 - \dfrac{5000}{x^2}$.

9. a. $R'(x) = \dfrac{d}{dx}\left(8000x - 100x^2\right) = 8000 - 200x$.

b. $R'(39) = 8000 - 200(39) = 200$, $R'(40) = 8000 - 200(40) = 0$, and $R'(41) = 8000 - 200(41) = -200$.

c. This suggests the total revenue is maximized if the price charged per passenger is \$40.

10. a. $R(x) = px = x(-0.04x + 800) = -0.04x^2 + 800x$.

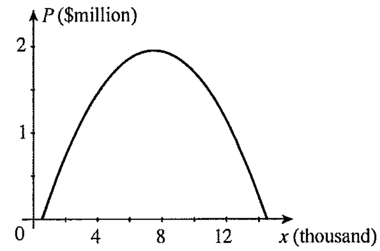
b. $R'(x) = -0.08x + 800$.

c. $R'(5000) = -0.08(5000) + 800 = 400$. This says that when the level of production is 5000 units, the production of the next speaker system will bring an additional revenue of \$400.

11. a. $P(x) = R(x) - C(x) = \left(-0.04x^2 + 800x\right) - (200x + 300{,}000) = -0.04x^2 + 600x - 300{,}000$.

b. $P'(x) = -0.08x + 600$.

c. $P'(5000) = -0.08(5000) + 600 = 200$ and $P'(8000) = -0.08(8000) + 600 = -40$.

d. 

The profit realized by the company increases as production increases, peaking at a production level of 7500 units. Beyond this level of production, the profit begins to fall.

12. a. $P(x) = -10x^2 + 1760x - 50{,}000$. To find the actual profit realized from renting the 51st unit, assuming that 50 units have already been rented, we calculate

$$P(51) - P(50) = \left[-10(51)^2 + 1760(51) - 50{,}000\right] - \left[-10(50)^2 + 1760(50) - 50{,}000\right]$$
$$= -26{,}010 + 89{,}760 - 50{,}000 + 25{,}000 - 88{,}000 + 50{,}000 = 750, \text{ or } \$750.$$

b. The marginal profit is given by $P'(x) = -20x + 1760$. When $x = 50$, $P'(50) = -20(50) + 1760 = 760$, or \$760.

13. a. The revenue function is $R(x) = px = (600 - 0.05x)x = 600x - 0.05x^2$ and the profit function is

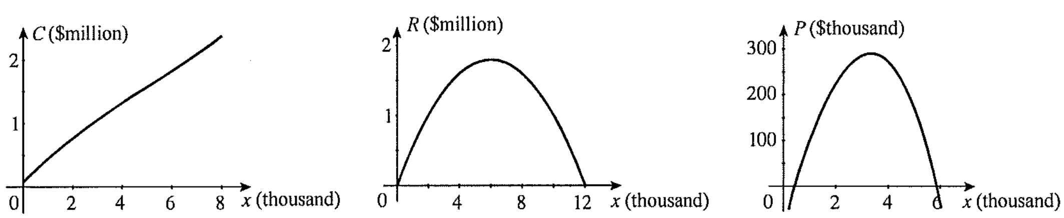
$$P(x) = R(x) - C(x) = \left(600x - 0.05x^2\right) - \left(0.000002x^3 - 0.03x^2 + 400x + 80{,}000\right)$$
$$= -0.000002x^3 - 0.02x^2 + 200x - 80{,}000.$$

b. $C'(x) = \dfrac{d}{dx}\left(0.000002x^3 - 0.03x^2 + 400x + 80{,}000\right) = 0.000006x^2 - 0.06x + 400$,

$R'(x) = \dfrac{d}{dx}\left(600x - 0.05x^2\right) = 600 - 0.1x$, and

$P'(x) = \dfrac{d}{dx}\left(-0.000002x^3 - 0.02x^2 + 200x - 80{,}000\right) = -0.000006x^2 - 0.04x + 200.$

c. $C'(2000) = 0.000006(2000)^2 - 0.06(2000) + 400 = 304$, and this says that at a production level of 2000 units, the cost for producing the 2001st unit is \$304. $R'(2000) = 600 - 0.1(2000) = 400$, and this says that the revenue realized in selling the 2001st unit is \$400. $P'(2000) = R'(2000) - C'(2000) = 400 - 304 = 96$, and this says that the revenue realized in selling the 2001st unit is \$96.

d. 

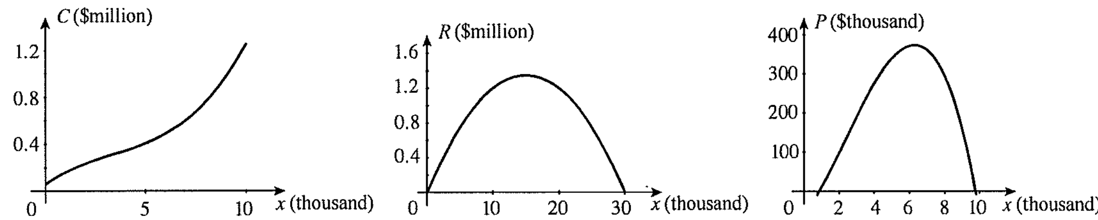
14. a. $R(x) = xp(x) = -0.006x^2 + 180x$ and

$$P(x) = R(x) - C(x) = \left(-0.006x^2 + 180x\right) - \left(0.000002x^3 - 0.02x^2 + 120x + 60{,}000\right)$$
$$= -0.000002x^3 + 0.014x^2 + 60x - 60{,}000.$$

b. $C'(x) = 0.000006x^2 - 0.04x + 120$, $R'(x) = -0.012x + 180$, and $P'(x) = -0.000006x^2 + 0.028x + 60$.

c. $C'(2000) = 0.000006(2000)^2 - 0.04(2000) + 120 = 64$, $R'(2000) = -0.012(2000) + 180 = 156$, and $P'(2000) = -0.000006(2000)^2 + 0.028(2000) + 60 = 92$.

d.



15. a. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{0.000002x^3 - 0.03x^2 + 400x + 80{,}000}{x} = 0.000002x^2 - 0.03x + 400 + \dfrac{80{,}000}{x}$.

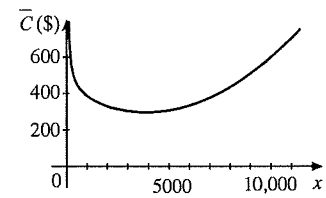
b. $\overline{C}'(x) = 0.000004x - 0.03 - \dfrac{80{,}000}{x^2}$.

c. $\overline{C}'(5000) = 0.000004(5000) - 0.03 - \dfrac{80{,}000}{5000^2} \approx -0.0132$, and this says that at a production level of 5000 units, the average cost of production is dropping at the rate of approximately a penny per unit.

$\overline{C}'(10{,}000) = 0.000004(10000) - 0.03 - \dfrac{80{,}000}{10{,}000^2} \approx 0.0092$,

and this says that, at a production level of 10,000 units, the average cost of production is increasing at the rate of approximately a penny per unit.

d.



16. a. $C(x) = 0.000002x^3 - 0.02x^2 + 120x + 60{,}000$, so $\overline{C}(x) = 0.000002x^2 - 0.02x + 120 + \dfrac{60{,}000}{x}$.

b. The marginal average cost function is given by $\overline{C}'(x) = 0.000004x - 0.02 - \dfrac{60{,}000}{x^2}$.

c. $\overline{C}'(5000) = 0.000004(5000) - 0.02 - \dfrac{60{,}000}{(5000)^2}$

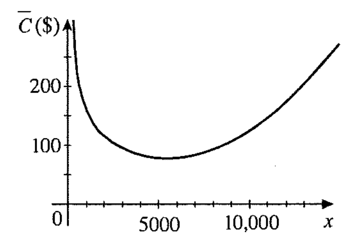
$= 0.02 - 0.02 - 0.0024 = -0.0024$ and

$\overline{C}'(10{,}000) = 0.000004(10{,}000) - 0.02 - \dfrac{60{,}000}{(10000)^2}$

$= 0.04 - 0.02 - 0.0006 = 0.0194$.

We conclude that the average cost is decreasing when 5000 TV sets are produced and increasing when 10,000 units are produced.

d.



17. a. $R(x) = px = \dfrac{50x}{0.01x^2 + 1}$.

b. $R'(x) = \dfrac{(0.01x^2 + 1)\,50 - 50x(0.02x)}{(0.01x^2 + 1)^2} = \dfrac{50 - 0.5x^2}{(0.01x^2 + 1)^2}$.

c. $R'(2) = \dfrac{50 - 0.5(4)}{[0.01(4) + 1]^2} \approx 44.379$. This result says that at a sales level of 2000 units, the revenue increases at the rate of approximately \$44,379 per 1000 units.

18. $\dfrac{dC}{dx} = \dfrac{d}{dx}(0.712x + 95.05) = 0.712$.

19. $C(x) = 0.873x^{1.1} + 20.34$, so $C'(x) = 0.873(1.1)x^{0.1}$. $C'(10) = 0.873(1.1)(10)^{0.1} = 1.21$.