## 3.3 The Chain Rule

### Concept Questions page 194

1. The derivative of $h(x) = g(f(x))$ is equal to the derivative of $g$ evaluated at $f(x)$ times the derivative of $f$.

2. $h'(x) = \dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$.

3. $(g \circ f)'(t) = [(g \circ f)(t)]' = g'(f(t)) f'(t)$ describes the rate of change of the revenue as a function of time.

4. $(f \circ g)'(t)$ gives the rate of change of the air temperature.

### Exercises page 194

1. $f(x) = (2x-1)^3$, so $f'(x) = 3(2x-1)^2 \frac{d}{dx}(2x-1) = 3(2x-1)^2(2) = 6(2x-1)^2$.

2. $f(x) = (1-x)^4$, so $f'(x) = 4(1-x)^3(-1) = -4(1-x)^3$.

3. $f(x) = (x^2+2)^5$, so $f'(x) = 5(x^2+2)^4(2x) = 10x(x^2+2)^4$.

4. $f(t) = 2(t^3-1)^5$, so $f'(t) = (2)(5)(t^3-1)^4(3t^2) = 30t^2(t^3-1)^4$.

5. $f(x) = (2x-x^2)^3$, so $f'(x) = 3(2x-x^2)^2 \frac{d}{dx}(2x-x^2) = 3(2x-x^2)^2(2-2x) = 6x^2(1-x)(2-x)^2$.

6. $f(x) = 3(x^3-x)^4$, so $f'(x) = (3)(4)(x^3-x)^3(3x^2-1) = 12(3x^2-1)(x^3-x)^3$.

7. $f(x) = (2x+1)^{-2}$, so $f'(x) = -2(2x+1)^{-3} \frac{d}{dx}(2x+1) = -2(2x+1)^{-3}(2) = -4(2x+1)^{-3}$.

8. $f(t) = \frac{1}{2}(2t^2+t)^{-3}$, so $f'(t) = \frac{1}{2}(-3)(2t^2+t)^{-4}(4t+1) = -\dfrac{3(1+4t)}{2(2t^2+t)^4}$.

9. $f(x) = (x^2-4)^{5/2}$, so $f'(x) = \frac{5}{2}(x^2-4)^{3/2} \frac{d}{dx}(x^2-4) = \frac{5}{2}(x^2-4)^{3/2}(2x) = 5x(x^2-4)^{3/2}$.

10. $f(t) = (3t^2-2t+1)^{3/2}$, so $f'(t) = \frac{3}{2}(3t^2-2t+1)^{1/2}(6t-2) = 3(3t-1)(3t^2-2t+1)^{1/2}$.

11. $f(x) = \sqrt{3x-2} = (3x-2)^{1/2}$, so $f'(x) = \frac{1}{2}(3x-2)^{-1/2}(3) = \frac{3}{2}(3x-2)^{-1/2} = \dfrac{3}{2\sqrt{3x-2}}$.

12. $f(t) = \sqrt{3t^2-t} = (3t^2-t)^{1/2}$, so $f'(t) = \frac{1}{2}(3t^2-t)^{-1/2}(6t-1) = \dfrac{6t-1}{2\sqrt{3t^2-t}}$.

13. $f(x) = \sqrt[3]{1-x^2}$, so

$$f'(x) = \tfrac{d}{dx}(1-x^2)^{1/3} = \tfrac{1}{3}(1-x^2)^{-2/3}\tfrac{d}{dx}(1-x^2) = \tfrac{1}{3}(1-x^2)^{-2/3}(-2x) = -\tfrac{2}{3}x(1-x^2)^{-2/3}$$
$$= \frac{-2x}{3(1-x^2)^{2/3}}.$$

14. $f(x) = \sqrt{2x^2-2x+3}$, so $f'(x) = \frac{1}{2}(2x^2-2x+3)^{-1/2}(4x-2) = (2x-1)(2x^2-2x+3)^{-1/2}$.

**15.** $f(x) = \dfrac{1}{(2x+3)^3} = (2x+3)^{-3}$, so $f'(x) = -3(2x+3)^{-4}(2) = -6(2x+3)^{-4} = -\dfrac{6}{(2x+3)^4}$.

**16.** $f(x) = \dfrac{2}{(x^2-1)^4}$, so $f'(x) = 2\frac{d}{dx}(x^2-1)^{-4} = 2(-4)(x^2-1)^{-5}(2x) = -16x(x^2-1)^{-5}$.

**17.** $f(t) = \dfrac{1}{\sqrt{2t-4}}$, so $f'(t) = \frac{d}{dt}(2t-4)^{-1/2} = -\frac{1}{2}(2t-4)^{-3/2}(2) = -(2t-4)^{-3/2} = -\dfrac{1}{(2t-4)^{3/2}}$.

**18.** $f(x) = \dfrac{1}{\sqrt{2x^2-1}} = (2x^2-1)^{-1/2}$, so $f'(x) = -\frac{1}{2}(2x^2-1)^{-3/2}(4x) = -\dfrac{2x}{\sqrt{(2x^2-1)^3}}$.

**19.** $y = \dfrac{1}{(4x^4+x)^{3/2}}$, so $\frac{dy}{dx} = \frac{d}{dx}(4x^4+x)^{-3/2} = -\frac{3}{2}(4x^4+x)^{-5/2}(16x^3+1) = -\frac{3}{2}(16x^3+1)(4x^4+x)^{-5/2}$.

**20.** $f(t) = \dfrac{4}{\sqrt[3]{2t^2+t}}$, so $f'(t) = 4\frac{d}{dt}(2t^2+t)^{-1/3} = -\frac{4}{3}(2t^2+t)^{-4/3}(4t+1) = -\frac{4}{3}(4t+1)(2t^2+t)^{-4/3}$.

**21.** $f(x) = (3x^2+2x+1)^{-2}$, so

$$f'(x) = -2(3x^2+2x+1)^{-3}\frac{d}{dx}(3x^2+2x+1) = -2(3x^2+2x+1)^{-3}(6x+2)$$
$$= -4(3x+1)(3x^2+2x+1)^{-3}.$$

**22.** $f(t) = (5t^3+2t^2-t+4)^{-3}$, so $f'(t) = -3(5t^3+2t^2-t+4)^{-4}(15t^2+4t-1)$.

**23.** $f(x) = (x^2+1)^3 - (x^3+1)^2$, so

$$f'(x) = 3(x^2+1)^2\frac{d}{dx}(x^2+1) - 2(x^3+1)\frac{d}{dx}(x^3+1) = 3(x^2+1)^2(2x) - 2(x^3+1)(3x^2)$$
$$= 6x\left[(x^2+1)^2 - x(x^3+1)\right] = 6x(2x^2-x+1).$$

**24.** $f(t) = (2t-1)^4 + (2t+1)^4$, so $f'(t) = 4(2t-1)^3(2) + 4(2t+1)^3(2) = 8\left[(2t-1)^3 + (2t+1)^3\right]$.

**25.** $f(t) = (t^{-1}-t^{-2})^3$, so $f'(t) = 3(t^{-1}-t^{-2})^2\frac{d}{dt}(t^{-1}-t^{-2}) = 3(t^{-1}-t^{-2})^2(-t^{-2}+2t^{-3})$.

**26.** $f(v) = (v^{-3}+4v^{-2})^3$, so $f'(v) = 3(v^{-3}+4v^{-2})^2(-3v^{-4}-8v^{-3})$.

**27.** $f(x) = \sqrt{x+1} + \sqrt{x-1} = (x+1)^{1/2} + (x-1)^{1/2}$, so

$$f'(x) = \tfrac{1}{2}(x+1)^{-1/2}(1) + \tfrac{1}{2}(x-1)^{-1/2}(1) = \tfrac{1}{2}\left[(x+1)^{-1/2} + (x-1)^{-1/2}\right].$$

**28.** $f(u) = (2u+1)^{3/2} + (u^2-1)^{-3/2}$, so

$$f'(u) = \tfrac{3}{2}(2u+1)^{1/2}(2) - \tfrac{3}{2}(u^2-1)^{-5/2}(2u) = 3(2u+1)^{1/2} - 3u(u^2-1)^{-5/2}.$$

**29.** $f(x) = 2x^2(3-4x)^4$, so

$$f'(x) = 2x^2(4)(3-4x)^3(-4) + (3-4x)^4(4x) = 4x(3-4x)^3(-8x+3-4x)$$
$$= 4x(3-4x)^3(-12x+3) = (-12x)(4x-1)(3-4x)^3.$$

**30.** $h(t) = t^2(3t+4)^3$, so

$$h'(t) = 2t(3t+4)^3 + t^2(3)(3t+4)^2(3) = t(3t+4)^2[2(3t+4)+9t] = t(15t+8)(3t+4)^2.$$

**31.** $f(x) = (x-1)^2(2x+1)^4$, so

$$\begin{aligned} f'(x) &= (x-1)^2 \tfrac{d}{dx}(2x+1)^4 + (2x+1)^4 \tfrac{d}{dx}(x-1)^2 \quad \text{(by the Product Rule)} \\ &= (x-1)^2(4)(2x+1)^3 \tfrac{d}{dx}(2x+1) + (2x+1)^4(2)(x-1)\tfrac{d}{dx}(x-1) \\ &= 8(x-1)^2(2x+1)^3 + 2(x-1)(2x+1)^4 = 2(x-1)(2x+1)^3(4x-4+2x+1) \\ &= 6(x-1)(2x-1)(2x+1)^3. \end{aligned}$$

**32.** $g(u) = (u+1)^{1/2}(1-2u)^8$, so

$$\begin{aligned} g'(u) &= (u+1)^{1/2}(8)\left(1-2u^2\right)^7(-4u) + \left(1-2u^2\right)^8\left(\tfrac{1}{2}\right)(u+1)^{-1/2} \\ &= -\tfrac{1}{2}(u+1)^{-1/2}\left(1-2u^2\right)^7\left[64u(u+1) - \left(1-2u^2\right)\right] \\ &= -\frac{\left(66u^2+64u-1\right)\left(1-2u^2\right)^7}{2\sqrt{u+1}} = \frac{\left(2u^2-1\right)^7\left(66u^2+64u-1\right)}{2\sqrt{u+1}}. \end{aligned}$$

**33.** $f(x) = \left(\dfrac{x+3}{x-2}\right)^3$, so

$$\begin{aligned} f'(x) &= 3\left(\frac{x+3}{x-2}\right)^2 \frac{d}{dx}\left(\frac{x-3}{x-2}\right) = 3\left(\frac{x+3}{x-2}\right)^2\left[\frac{(x-2)(1)-(x+3)(1)}{(x-2)^2}\right] \\ &= 3\left(\frac{x+3}{x-2}\right)^2\left[-\frac{5}{(x-2)^2}\right] = -\frac{15(x+3)^2}{(x-2)^4}. \end{aligned}$$

**34.** $f(x) = \left(\dfrac{x+1}{x-1}\right)^5$, so $f'(x) = 5\left(\dfrac{x+1}{x-1}\right)^4\left[\dfrac{(x-1)(1)-(x+1)(1)}{(x-1)^2}\right] = -\dfrac{10(x+1)^4}{(x-1)^6}$.

**35.** $s(t) = \left(\dfrac{t}{2t+1}\right)^{3/2}$, so

$$\begin{aligned} s'(t) &= \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2}\frac{d}{dt}\left(\frac{t}{2t+1}\right) = \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2}\left[\frac{(2t+1)(1)-t(2)}{(2t+1)^2}\right] \\ &= \frac{3}{2}\left(\frac{t}{2t+1}\right)^{1/2}\left[\frac{1}{(2t+1)^2}\right] = \frac{3t^{1/2}}{2(2t+1)^{5/2}}. \end{aligned}$$

**36.** $g(s) = \left(s^2+\dfrac{1}{s}\right)^{3/2} = \left(s^2+s^{-1}\right)^{3/2}$, so

$$g'(s) = \tfrac{3}{2}\left(s^2+s^{-1}\right)^{1/2}\left(2s-s^{-2}\right) = \frac{3}{2}\left(s^2+\frac{1}{s}\right)^{1/2}\left(2s-\frac{1}{s^2}\right) = \frac{3}{2}\left(\frac{s^3+1}{s}\right)^{1/2}\left(\frac{2s^3-1}{s^2}\right).$$

**37.** $g(u) = \left(\dfrac{u+1}{3u+2}\right)^{1/2}$, so

$$\begin{aligned} g'(u) &= \frac{1}{2}\left(\frac{u+1}{3u+2}\right)^{-1/2}\frac{d}{du}\left(\frac{u+1}{3u+2}\right) = \frac{1}{2}\left(\frac{u+1}{3u+2}\right)^{-1/2}\left[\frac{(3u+2)(1)-(u+1)(3)}{(3u+2)^2}\right] \\ &= -\frac{1}{2\sqrt{u+1}\,(3u+2)^{3/2}}. \end{aligned}$$

**38.** $g(x) = \left(\frac{2x+1}{2x-1}\right)^{1/2}$, so

$$g'(x) = \frac{1}{2}\left(\frac{2x+1}{2x-1}\right)^{-1/2}\left[\frac{(2x-1)(2)-(2x+1)(2)}{(2x-1)^2}\right] = \frac{1}{2}\left(\frac{2x+1}{2x-1}\right)^{-1/2}\left(-\frac{4}{(2x-1)^2}\right)$$
$$= -\frac{2}{(2x+1)^{1/2}(2x-1)^{3/2}}.$$

**39.** $f(x) = \frac{x^2}{(x^2-1)^4}$, so

$$f'(x) = \frac{(x^2-1)^4 \frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(x^2-1)^4}{\left[(x^2-1)^4\right]^2} = \frac{(x^2-1)^4(2x) - x^2(4)(x^2-1)^3(2x)}{(x^2-1)^8}$$
$$= \frac{(x^2-1)^3(2x)(x^2-1-4x^2)}{(x^2-1)^8} = \frac{(-2x)(3x^2+1)}{(x^2-1)^5}.$$

**40.** $g(u) = \frac{2u^2}{(u^2+u)^3}$, so

$$g'(u) = \frac{(u^2+u)^3(4u) - (2u^2)3(u^2+u)^2(2u+1)}{(u^2+u)^6}$$
$$= \frac{2u(u^2+u)^2[2(u^2+u) - 3u(2u+1)]}{(u^2+u)^6} = \frac{-2u(4u^2+u)}{(u^2+u)^4} = \frac{-2u^2(4u+1)}{u^4(u+1)^4} = -\frac{2(4u+1)}{u^2(u+1)^4}.$$

**41.** $h(x) = \frac{(3x^2+1)^3}{(x^2-1)^4}$, so

$$h'(x) = \frac{(x^2-1)^4(3)(3x^2+1)^2(6x) - (3x^2+1)^3(4)(x^2-1)^3(2x)}{(x^2-1)^8}$$
$$= \frac{2x(x^2-1)^3(3x^2+1)^2[9(x^2-1) - 4(3x^2+1)]}{(x^2-1)^8} = -\frac{2x(3x^2+13)(3x^2+1)^2}{(x^2-1)^5}.$$

**42.** $g(t) = \frac{(2t-1)^2}{(3t+2)^4}$, so

$$g'(t) = \frac{(3t+2)^4(2)(2t-1)(2) - (2t-1)^2(4)(3t+2)^3(3)}{(3t+2)^8}$$
$$= \frac{4(3t+2)^3(2t-1)[(3t+2) - 3(2t-1)]}{(3t+2)^8} = \frac{4(2t-1)(5-3t)}{(3t+2)^5}.$$

**43.** $f(x) = \frac{\sqrt{2x+1}}{x^2-1}$, so

$$f'(x) = \frac{(x^2-1)\left(\frac{1}{2}\right)(2x+1)^{-1/2}(2) - (2x+1)^{1/2}(2x)}{(x^2-1)^2} = \frac{(2x+1)^{-1/2}\left[(x^2-1) - (2x+1)(2x)\right]}{(x^2-1)^2}$$
$$= -\frac{3x^2+2x+1}{\sqrt{2x+1}(x^2-1)^2}.$$

**84.** $p = f(t) = 50\left(\frac{t^2+2t+4}{t^2+4t+8}\right)$ and $R(p) = 1000\left(\frac{p+4}{p+2}\right)$. We want $\frac{dR}{dt} = \frac{dR}{dp}\cdot\frac{dp}{dt}$. Now $\frac{dR}{dp} = 1000\left[\frac{(p+2)(1)-(p+4)(1)}{(p+2)^2}\right] = -\frac{2000}{(p+2)^2}$ and $\frac{dp}{dt} = 50\left[\frac{(t^2+4t+8)(2t+2)-(t^2+2t+4)(2t+4)}{(t^2+4t+8)^2}\right] = \frac{100t(t+4)}{(t^2+4t+8)^2}$. When $t=2$, $p = 50\left(\frac{4+4+4}{4+8+8}\right) = 30$, and $\frac{dR}{dt} = -\frac{2000}{(p+2)^2}\cdot\frac{100t(t+4)}{(t^2+4t+8)^2}\bigg|_{t=2} = -\frac{2000}{(32)^2}\cdot\frac{100(2)(6)}{(4+8+8)^2} \approx -5.86$; that is, the passage will decrease at the rate of approximately \$5.86 per passenger per year.

**85.** True. This is just the statement of the Chain Rule.

**86.** True. $\frac{d}{dx}[f(cx)] = f'(cx)\frac{d}{dx}(cx) = f'(cx)\cdot c$.

**87.** True. $\frac{d}{dx}\sqrt{f(x)} = \frac{d}{dx}[f(x)]^{1/2} = \frac{1}{2}[f(x)]^{-1/2}f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$.

**88.** False. Let $f(x) = x$. Then $f\left(\frac{1}{x}\right) = \frac{1}{x}$ and so $f'(x) = -\frac{1}{x^2}$. But $f'(x) = 1$, so $f'\left(\frac{1}{x}\right) = 1$.

**89.** Let $f(x) = x^{1/n}$ so that $[f(x)]^n = x$. Differentiating both sides with respect to $x$, we get $n[f(x)]^{n-1}f'(x) = 1$, so $f'(x) = \frac{1}{n[f(x)]^{n-1}} = \frac{1}{n[x^{1/n}]^{n-1}} = \frac{1}{nx^{1-(1/n)}} = \frac{1}{n}x^{(1/n)-1}$, as was to be shown.

**90.** Let $f(x) = x^r = x^{m/n} = (x^m)^{1/n}$. Then $[f(x)]^n = x^m$. Therefore,

$$n[f(x)]^{n-1}f'(x) = \frac{m}{n}[f(x)]^{-n+1}x^{m-1} = \frac{m}{n}\left(x^{m/n}\right)^{-n+1}x^{m-1} = \frac{m}{n}x^{[m(-n+1)/n]+m-1}$$
$$= \frac{m}{n}x^{(m-n)/n} = \frac{m}{n}x^{(m/n)-1} = rx^{r-1}.$$

## Using Technology page 198

**1.** 0.5774 **2.** 1.4364 **3.** 0.9390

**4.** 3.9051 **5.** −4.9498 **6.** 0.1056

**7.** **a.** Using the numerical derivative operation, we find that $N'(0) = 5.41450$, so the rate of change of the number of people watching TV on mobile phones at the beginning of 2007 is approximately 5.415 million/year.

**b.** $N'(4) \approx 2.5136$, so the corresponding rate of change at the beginning of 2011 is expected to be approximately 2.5136 million/year.

**8.** **a.** 43.6 million **b.** 0.432745 million/year