3.2 The Product and Quotient Rules

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1. a. The derivative of the product of two functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function.

   b. The derivative of the quotient of two functions is equal to the quotient whose numerator is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator and whose denominator is the square of the denominator of the quotient.

2. a. $h'(x) = f(x)g'(x) + f'(x)g(x)$, so $h'(1) = f(1)g'(1) + f'(1)g(1) = (3)(4) + (-1)(2) = 10$.

   b. $F'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, so $F'(1) = \dfrac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} = \dfrac{2(-1) - 3(4)}{2^2} = -\dfrac{7}{2}$.

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1. $f(x) = 2x(x^2+1)$, so $f'(x) = 2x\frac{d}{dx}(x^2+1) + (x^2+1)\frac{d}{dx}(2x) = 2x(2x) + (x^2+1)(2) = 6x^2 + 2$.

2. $f(x) = 3x^2(x-1)$, so $f'(x) = 3x^2\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}(3x^2) = 3x^2 + (x-1)(6x) = 9x^2 - 6x$.

3. $f(t) = (t-1)(2t+1)$, so
   $f'(t) = (t-1)\frac{d}{dt}(2t+1) + (2t+1)\frac{d}{dt}(t-1) = (t-1)(2) + (2t+1)(1) = 4t - 1$.

4. $f(x) = (2x+3)(3x-4)$, so
   $f'(x) = (2x+3)\frac{d}{dx}(3x-4) + (3x-4)\frac{d}{dx}(2x+3) = (2x+3)(3) + (3x-4)(2) = 12x + 1$.

5. $f(x) = (3x+1)(x^2-2)$, so
   $f'(x) = (3x+1)\frac{d}{dx}(x^2-2) + (x^2-2)\frac{d}{dx}(3x+1) = (3x+1)(2x) + (x^2-2)(3) = 9x^2 + 2x - 6$.

6. $f(x) = (x+1)(2x^2-3x+1)$, so
   $$f'(x) = (x+1)\frac{d}{dx}(2x^2-3x+1) + (2x^2-3x+1)\frac{d}{dx}(x+1) = (x+1)(4x-3) + (2x^2-3x+1)(1)$$
   $$= 4x^2 - 3x + 4x - 3 + 2x^2 - 3x + 1 = 6x^2 - 2x - 2 = 2(3x^2 - x - 1).$$

7. $f(x) = (x^3-1)(x+1)$, so
   $f'(x) = (x^3-1)\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}(x^3-1) = (x^3-1)(1) + (x+1)(3x^2) = 4x^3 + 3x^2 - 1$.

8. $f(x) = (x^3-12x)(3x^2+2x)$, so
   $$f'(x) = (x^3-12x)\frac{d}{dx}(3x^2+2x) + (3x^2+2x)\frac{d}{dx}(x^3-12x)$$
   $$= (x^3-12x)(6x+2) + (3x^2+2x)(3x^2-12)$$
   $$= 6x^4 + 2x^3 - 72x^2 - 24x + 9x^4 + 6x^3 - 36x^2 - 24x = 15x^4 + 8x^3 - 108x^2 - 48x.$$

9. $f(w) = \left(w^3 - w^2 + w - 1\right)\left(w^2+2\right)$, so

$$f'(w) = \left(w^3 - w^2 + w - 1\right)\frac{d}{dw}\left(w^2+2\right) + \left(w^2+2\right)\frac{d}{dw}\left(w^3 - w^2 + w - 1\right)$$
$$= \left(w^3 - w^2 + w - 1\right)(2w) + \left(w^2+2\right)\left(3w^2 - 2w + 1\right)$$
$$= 2w^4 - 2w^3 + 2w^2 - 2w + 3w^4 - 2w^3 + w^2 + 6w^2 - 4w + 2 = 5w^4 - 4w^3 + 9w^2 - 6w + 2.$$

10. $f(x) = \frac{1}{5}x^5 + \left(x^2+1\right)\left(x^2 - x - 1\right) + 28$, so

$$f'(x) = x^4 + \left(x^2+1\right)(2x-1) + 2x\left(x^2 - x - 1\right) = x^4 + 2x^3 - x^2 + 2x - 1 + 2x^3 - 2x^2 - 2x$$
$$= x^4 + 4x^3 - 3x^2 - 1.$$

11. $f(x) = \left(5x^2+1\right)\left(2\sqrt{x} - 1\right)$, so

$$f'(x) = \left(5x^2+1\right)\tfrac{d}{dx}\left(2x^{1/2} - 1\right) + \left(2x^{1/2} - 1\right)\tfrac{d}{dx}\left(5x^2+1\right) = \left(5x^2+1\right)\left(x^{-1/2}\right) + \left(2x^{1/2} - 1\right)(10x)$$
$$= 5x^{3/2} + x^{-1/2} + 20x^{3/2} - 10x = \frac{25x^2 - 10x\sqrt{x} + 1}{\sqrt{x}}.$$

12. $f(t) = \left(1 + \sqrt{t}\right)\left(2t^2 - 3\right)$, so

$$f'(t) = \left(1 + t^{1/2}\right)(4t) + \left(2t^2 - 3\right)\left(\tfrac{1}{2}t^{-1/2}\right) = 4t + 4t^{3/2} + t^{3/2} - \tfrac{3}{2}t^{-1/2} = 5t^{3/2} + 4t - \tfrac{3}{2}t^{-1/2}$$
$$= \frac{10t^2 + 8t\sqrt{t} - 3}{2\sqrt{t}}.$$

13. $f(x) = \left(x^2 - 5x + 2\right)\left(x - \dfrac{2}{x}\right)$, so

$$f'(x) = \left(x^2 - 5x + 2\right)\frac{d}{dx}\left(x - \frac{2}{x}\right) + \left(x - \frac{2}{x}\right)\frac{d}{dx}\left(x^2 - 5x + 2\right)$$
$$= \frac{\left(x^2 - 5x + 2\right)\left(x^2+2\right)}{x^2} + \frac{\left(x^2-2\right)(2x-5)}{x} = \frac{\left(x^2 - 5x + 2\right)\left(x^2+2\right) + x\left(x^2-2\right)(2x-5)}{x^2}$$
$$= \frac{x^4 + 2x^2 - 5x^3 - 10x + 2x^2 + 4 + 2x^4 - 5x^3 - 4x^2 + 10x}{x^2} = \frac{3x^4 - 10x^3 + 4}{x^2}.$$

14. $f(x) = \left(x^3 + 2x + 1\right)\left(2 + \dfrac{1}{x^2}\right) = 2x^3 + 4x + 2 + x + \dfrac{2}{x} + \dfrac{1}{x^2}$, so

$$f'(x) = \frac{d}{dx}\left(2x^3 + 5x + 2 + \frac{2}{x} + \frac{1}{x^2}\right) = 6x^2 + 5 - \frac{2}{x^2} - \frac{2}{x^3} = \frac{6x^5 + 5x^3 - 2x - 2}{x^3}.$$

15. $f(x) = \dfrac{1}{x-2}$, so $f'(x) = \dfrac{(x-2)\frac{d}{dx}(1) - (1)\frac{d}{dx}(x-2)}{(x-2)^2} = \dfrac{0 - 1(1)}{(x-2)^2} = -\dfrac{1}{(x-2)^2}.$

16. $g(x) = \dfrac{3}{2x+4} + 2x^2$, so $g'(x) = \dfrac{d}{dx}\left(\dfrac{3}{2x+4}\right) + \dfrac{d}{dx}\left(2x^2\right) = \dfrac{(2x+4)(0) - 3(2)}{(2x+4)^2} + 4x = -\dfrac{6}{(2x+4)^2} + 4x.$

17. $f(x) = \dfrac{2x-1}{2x+1}$, so

$$f'(x) = \frac{(2x+1)\frac{d}{dx}(2x-1) - (2x-1)\frac{d}{dx}(2x+1)}{(2x+1)^2} = \frac{2(2x+1) - (2x-1)(2)}{(2x+1)^2} = \frac{4}{(2x+1)^2}.$$

18. $f(t) = \dfrac{1-2t}{1+3t}$, so $f'(t) = \dfrac{(1+3t)(-2)-(1-2t)(3)}{(1+3t)^2} = \dfrac{-5}{(1+3t)^2}$.

19. $f(x) = \dfrac{1}{x^2+x+2}$, so $f'(x) = \dfrac{\left(x^2+x+2\right)(0)-(1)(2x+1)}{\left(x^2+x+2\right)^2} = -\dfrac{2x+1}{\left(x^2+x+2\right)^2}$.

20. $f(u) = \dfrac{u}{u^2+1}$, so $f'(u) = \dfrac{\left(u^2+1\right)\dfrac{d}{du}(u) - u\dfrac{d}{du}\left(u^2+1\right)}{\left(u^2+1\right)^2} = \dfrac{\left(u^2+1\right)(1)-u(2u)}{\left(u^2+1\right)^2} = \dfrac{1-u^2}{\left(u^2+1\right)^2}$.

21. $f(s) = \dfrac{s^2-4}{s+1}$, so

$$f'(s) = \frac{(s+1)\dfrac{d}{ds}\left(s^2-4\right)-\left(s^2-4\right)\dfrac{d}{ds}(s+1)}{(s+1)^2} = \frac{(s+1)(2s)-\left(s^2-4\right)(1)}{(s+1)^2} = \frac{s^2+2s+4}{(s+1)^2}.$$

22. $f(x) = \dfrac{x^3-2}{x^2+1}$, so

$$f'(x) = \frac{\left(x^2+1\right)\frac{d}{dx}\left(x^3-2\right)-\left(x^3-2\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} = \frac{\left(x^2+1\right)\left(3x^2\right)-\left(x^3-2\right)(2x)}{\left(x^2+1\right)^2} = \frac{x\left(x^3+3x+4\right)}{\left(x^2+1\right)^2}.$$

23. $f(x) = \dfrac{\sqrt{x}+1}{x^2+1}$, so

$$f'(x) = \frac{\left(x^2+1\right)\frac{d}{dx}\left(x^{1/2}\right)-\left(x^{1/2}+1\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2} = \frac{\left(x^2+1\right)\left(\frac{1}{2}x^{-1/2}\right)-\left(x^{1/2}+1\right)(2x)}{\left(x^2+1\right)^2}$$

$$= \frac{\left(\frac{1}{2}x^{-1/2}\right)\left[\left(x^2+1\right)-\left(x^{1/2}+1\right)4x^{3/2}\right]}{\left(x^2+1\right)^2} = \frac{1-3x^2-4x^{3/2}}{2\sqrt{x}\left(x^2+1\right)^2}.$$

24. $f(x) = \dfrac{x}{\sqrt{x}+2} = \dfrac{x}{x^{1/2}+2}$, so

$$f'(x) = \frac{\left(x^{1/2}+2\right)(1)-x\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}+2\right)^2} = \frac{x^{1/2}+2-\frac{1}{2}x^{1/2}}{\left(x^{1/2}+2\right)^2} = \frac{\frac{1}{2}x^{1/2}+2}{\left(x^{1/2}+2\right)^2} = \frac{\frac{1}{2}\left(x^{1/2}+4\right)}{\left(x^{1/2}+2\right)^2} = \frac{\sqrt{x}+4}{2\left(\sqrt{x}+2\right)^2}.$$

25. $f(x) = \dfrac{x^2+2}{x^2+x+1}$, so

$$f'(x) = \frac{\left(x^2+x+1\right)\frac{d}{dx}\left(x^2+2\right)-\left(x^2+2\right)\frac{d}{dx}\left(x^2+x+1\right)}{\left(x^2+x+1\right)^2}$$

$$= \frac{\left(x^2+x+1\right)(2x)-\left(x^2+2\right)(2x+1)}{\left(x^2+x+1\right)^2} = \frac{2x^3+2x^2+2x-2x^3-x^2-4x-2}{\left(x^2+x+1\right)^2} = \frac{x^2-2x-2}{\left(x^2+x+1\right)^2}.$$

26. $f(x) = \dfrac{x+1}{2x^2+2x+3}$, so

$$f'(x) = \frac{\left(2x^2+2x+3\right)(1)-(x+1)(4x+2)}{\left(2x^2+2x+3\right)^2} = \frac{2x^2+2x+3-4x^2-2x-4x-2}{\left(2x^2+2x+3\right)^2} = \frac{-2x^2-4x+1}{\left(2x^2+2x+3\right)^2}.$$

27. $f(x) = \dfrac{(x+1)(x^2+1)}{x-2} = \dfrac{(x^3+x^2+x+1)}{x-2}$, so

$$f'(x) = \frac{(x-2)\frac{d}{dx}(x^3+x^2+x+1) - (x^3+x^2+x+1)\frac{d}{dx}(x-2)}{(x-2)^2}$$
$$= \frac{(x-2)(3x^2+2x+1) - (x^3+x^2+x+1)}{(x-2)^2}$$
$$= \frac{3x^3+2x^2+x-6x^2-4x-2-x^3-x^2-x-1}{(x-2)^2} = \frac{2x^3-5x^2-4x-3}{(x-2)^2}.$$

28. $f(x) = (3x^2-1)\left(x^2 - \dfrac{1}{x}\right)$, so

$$f'(x) = 6x\left(x^2-\frac{1}{x}\right) + (3x^2-1)\left(2x+\frac{1}{x^2}\right) = 6x^3-6+6x^3+3-2x-\frac{1}{x^2} = 12x^3-2x-3-\frac{1}{x^2}.$$

29. $f(x) = \dfrac{x}{x^2-4} - \dfrac{x-1}{x^2+4} = \dfrac{x(x^2+4)-(x-1)(x^2-4)}{(x^2-4)(x^2+4)} = \dfrac{x^2+8x-4}{(x^2-4)(x^2+4)}$, so

$$f'(x) = \frac{(x^2-4)(x^2+4)\frac{d}{dx}(x^2+8x-4) - (x^2+8x-4)\frac{d}{dx}(x^4-16)}{(x^2-4)^2(x^2+4)^2}$$
$$= \frac{(x^2-4)(x^2+4)(2x+8) - (x^2+8x-4)(4x^3)}{(x^2-4)^2(x^2+4)^2}$$
$$= \frac{2x^5+8x^4-32x-128-4x^5-32x^4+16x^3}{(x^2-4)^2(x^2+4)^2} = \frac{-2x^5-24x^4+16x^3-32x-128}{(x^2-4)^2(x^2+4)^2}.$$

30. $f(x) = \dfrac{x+\sqrt{3x}}{3x-1}$, so

$$f'(x) = \frac{(3x-1)\left(1+\frac{1}{2}\sqrt{3}x^{-1/2}\right) - \left(x+\sqrt{3}x^{1/2}\right)(3)}{(3x-1)^2}$$
$$= \frac{3x+\frac{3}{2}\sqrt{3}x^{1/2}-1-\frac{1}{2}\sqrt{3}x^{-1/2}-3x-3\sqrt{3}x^{1/2}}{(3x-1)^2} = -\frac{3\sqrt{3}x+2\sqrt{x}+\sqrt{3}}{2\sqrt{x}(3x-1)^2}.$$

31. $h(x) = f(x)g(x)$, so $h'(x) = f(x)g'(x) + f'(x)g(x)$ by the Product Rule. Therefore, $h'(1) = f(1)g'(1) + f'(1)g(1) = (2)(3) + (-1)(-2) = 8$.

32. $h(x) = (x^2+1)g(x)$, so $h'(x) = (x^2+1)g'(x) + \frac{d}{dx}(x^2+1)\cdot g(x) = (x^2+1)g'(x) + 2xg(x)$. Therefore, $h'(1) = 2g'(1) + 2g(1) = (2)(3) + 2(-2) = 2$.

33. $h(x) = \dfrac{xf(x)}{x+g(x)}$. Using the Quotient Rule followed by the Product Rule, we obtain

$$h'(x) = \frac{[x+g(x)]\frac{d}{dx}[xf(x)] - xf(x)\frac{d}{dx}[x+g(x)]}{[x+g(x)]^2} = \frac{[x+g(x)][xf'(x)+f(x)] - xf(x)[1+g'(x)]}{[x+g(x)]^2}.$$

Therefore,

$$h'(1) = \frac{[1+g(1)][f'(1)+f(1)] - f(1)[1+g'(1)]}{[1+g(1)]^2} = \frac{(1-2)(-1+2)-2(1+3)}{(1-2)^2} = \frac{-1-8}{1} = -9.$$

68. a. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ with $q = 2.5$ and $p = 50$. Thus, $\frac{1}{f} = \frac{1}{2.5} + \frac{1}{50} = 0.42$, and so $f = \frac{1}{0.42} \approx 2.38$, or approximately 2.38 cm.

b. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, so $f = \dfrac{1}{\frac{1}{p} + \frac{1}{q}} = \frac{pq}{p+q}$. Differentiating with respect to p, we have

$\frac{df}{dp} = \frac{(p+q)q - pq(1)}{(p+q)^2} = \left(\frac{q}{p+q}\right)^2$. When $p = 50$, we have $\frac{df}{dp} = \left(\frac{2.5}{50+2.5}\right)^2 \approx 0.00227$, or 0.00227 cm/cm.

69. False. Take $f(x) = x$ and $g(x) = x$. Then $f(x)g(x) = x^2$, so $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}(x^2) = 2x \neq f'(x)g'(x) = 1$.

70. True. Using the Product Rule, $\frac{d}{dx}[xf(x)] = f(x)\frac{d}{dx}(x) + x\frac{d}{dx}[f(x)] = f(x)(1) + xf'(x)$.

71. False. Let $f(x) = x^3$. Then $\frac{d}{dx}\left[\frac{f(x)}{x^2}\right] = \frac{d}{dx}\left(\frac{x^3}{x^2}\right) = \frac{d}{dx}(x) = 1 \neq \frac{f'(x)}{2x} = \frac{3x^2}{2x} = \frac{3}{2}x$.

72. True. Using the Quotient Rule followed by the Product Rule,

$$\frac{d}{dx}\left[\frac{f(x)g(x)}{h(x)}\right] = \frac{h(x)\frac{d}{dx}[f(x)g(x)] - f(x)g(x)\frac{d}{dx}[h(x)]}{[h(x)]^2} = \frac{h(x)[f'(x)g(x) + f(x)g'(x)] - f(x)g(x)h'(x)}{[h(x)]^2}.$$

73. Let $f(x) = u(x)v(x)$ and $g(x) = w(x)$. Then $h(x) = f(x)g(x)$. Therefore, $h'(x) = f'(x)g(x) + f(x)g'(x)$. But $f'(x) = u(x)v'(x) + u'(x)v(x)$, so

$$h'(x) = [u(x)v'(x) + u'(x)v(x)]g(x) + u(x)v(x)w'(x) = u(x)v(x)w'(x) + u(x)v'(x)w(x) + u'(x)v(x)w(x).$$

74. Let $k(x) = \frac{f(x)}{g(x)}$.

a. $$\frac{k(x+h) - k(x)}{h} = \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}.$$

b. By adding $-f(x)g(x) + f(x)g(x)$ (which is equal to zero) to the numerator and simplifying, we have

$$\frac{k(x+h) - k(x)}{h} = \frac{1}{g(x+h)g(x)}\left\{\left[\frac{f(x+h) - f(x)}{h}\right]g(x) - \left[\frac{g(x+h) - g(x)}{h}\right]f(x)\right\}.$$

c. Taking the limit and using the definition of the derivative, we find

$$k'(x) = \lim_{h \to 0}\frac{k(x+h) - k(x)}{h} = \frac{1}{[g(x)]^2}[f'(x)g(x) - g'(x)f(x)] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

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1. 0.8750 **2.** 16.7980 **3.** 0.0774 **4.** −0.1314 **5.** −0.5000 **6.** 2.875

7. 31,312 per year **8. a.** 20,790 **b.** 554/year