3 DIFFERENTIATION

3.1 Basic Rules of Differentiation

Concept Questions page 169

1. **a.** The derivative of a constant is zero.

   b. The derivative of $f(x) = x^n$ is n times x raised to the $(n-1)$th power.

   c. The derivative of a constant times a function is the constant times the derivative of the function.

   d. The derivative of the sum is the sum of the derivatives.

2. **a.** $h'(x) = 2f'(x)$, so $h'(2) = 2f'(2) = 2(3) = 6$.

   b. $F'(x) = 3f'(x) - 4g'(x)$, so $F'(2) = 3f'(2) - 4g'(2) = 3(3) - 4(-2) = 17$.

3. **a.** $F'(x) = \dfrac{d}{dx}[af(x) + bg(x)] = \dfrac{d}{dx}[af(x)] + \dfrac{d}{dx}[bg(x)] = af'(x) + bg'(x)$.

   b. $F'(x) = \dfrac{d}{dx}\left[\dfrac{f(x)}{a}\right] = \dfrac{1}{a}\dfrac{d}{dx}[f(x)] = \dfrac{f'(x)}{a}$.

4. No. The expression on the left is the derivative at a of the function f, whereas the expression on the right is the derivative of the constant obtained by evaluating f at a. For example, if $f(x) = x^2$ and $a = 1$, then $[f'(x)](a) = 2x|_{x=1} = 2$, but $\dfrac{d}{dx}[f(a)] = \dfrac{d}{dx}(1^2) = 0$.

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1. $f'(x) = \frac{d}{dx}(-3) = 0$.

2. $f'(x) = \frac{d}{dx}(365) = 0$.

3. $f'(x) = \frac{d}{dx}(x^5) = 5x^4$.

4. $f'(x) = \frac{d}{dx}(x^7) = 7x^6$.

5. $f'(x) = \frac{d}{dx}(x^{3.1}) = 3.1x^{2.1}$.

6. $f'(x) = \frac{d}{dx}(x^{0.8}) = 0.8x^{-0.2}$.

7. $f'(x) = \frac{d}{dx}(3x^2) = 6x$.

8. $f'(x) = \frac{d}{dx}(-2x^3) = -6x^2$.

9. $f'(r) = \frac{d}{dr}(\pi r^2) = 2\pi r$.

10. $f'(r) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$.

11. $f'(x) = \frac{d}{dx}(9x^{1/3}) = \frac{1}{3}(9)x^{(1/3-1)} = 3x^{-2/3}$.

12. $f'(x) = \frac{d}{dx}\left(\frac{5}{4}x^{4/5}\right) = \left(\frac{4}{5}\right)\left(\frac{5}{4}\right)x^{-1/5} = x^{-1/5}$.

13. $f'(x) = \frac{d}{dx}(3\sqrt{x}) = \frac{d}{dx}(3x^{1/2}) = \frac{1}{2}(3)x^{-1/2} = \frac{3}{2}x^{-1/2} = \frac{3}{2\sqrt{x}}$.

14. $f'(u) = \frac{d}{du}\left(\frac{2}{\sqrt{u}}\right) = \frac{d}{du}(2u^{-1/2}) = -\frac{1}{2}(2)u^{-3/2} = -u^{-3/2}$.

15. $f'(x) = \frac{d}{dx}\left(7x^{-12}\right) = (-12)(7)x^{-12-1} = -84x^{-13}$.

16. $f'(x) = \frac{d}{dx}\left(0.3x^{-1.2}\right) = (0.3)(-1.2)x^{-2.2} = -0.36x^{-2.2}$.

17. $f'(x) = \frac{d}{dx}\left(5x^2 - 3x + 7\right) = 10x - 3$.

18. $f'(x) = \frac{d}{dx}\left(x^3 - 3x^2 + 1\right) = 3x^2 - 6x$.

19. $f'(x) = \frac{d}{dx}\left(-x^3 + 2x^2 - 6\right) = -3x^2 + 4x$.

20. $f'(x) = \frac{d}{dx}\left[\left(1 + 2x^2\right)^2 + 2x^3\right] = \frac{d}{dx}\left(1 + 4x^2 + 4x^4 + 2x^3\right) = 8x + 16x^3 + 6x^2 = 2x\left(8x^2 + 3x + 4\right)$.

21. $f'(x) = \frac{d}{dx}\left(0.03x^2 - 0.4x + 10\right) = 0.06x - 0.4$.

22. $f'(x) = \frac{d}{dx}\left(0.002x^3 - 0.05x^2 + 0.1x - 20\right) = 0.006x^2 - 0.1x + 0.1$.

23. $f(x) = \dfrac{2x^3 - 4x^2 + 3}{x} = 2x^2 - 4x + \dfrac{3}{x}$, so $f'(x) = \dfrac{d}{dx}\left(2x^2 - 4x + 3x^{-1}\right) = 4x - 4 - \dfrac{3}{x^2}$.

24. $f(x) = \dfrac{x^3 + 2x^2 + x - 1}{x} = x^2 + 2x + 1 - x^{-1}$, so $f'(x) = \frac{d}{dx}\left(x^2 + 2x + 1 - x^{-1}\right) = 2x + 2 + x^{-2}$.

25. $f'(x) = \frac{d}{dx}\left(4x^4 - 3x^{5/2} + 2\right) = 16x^3 - \frac{15}{2}x^{3/2}$.

26. $f'(x) = \frac{d}{dx}\left(5x^{4/3} - \frac{2}{3}x^{3/2} + x^2 - 3x + 1\right) = \frac{20}{3}x^{1/3} - x^{1/2} + 2x - 3$.

27. $f'(x) = \frac{d}{dx}\left(5x^{-1} + 4x^{-2}\right) = -5x^{-2} - 8x^{-3} = \dfrac{-5}{x^2} - \dfrac{8}{x^3}$.

28. $f'(x) = \frac{d}{dx}\left[-\frac{1}{3}\left(x^{-3} - x^6\right)\right] = -\frac{1}{3}\left(-3x^{-4} - 6x^5\right) = x^{-4} + 2x^5$.

29. $f'(t) = \frac{d}{dt}\left(4t^{-4} - 3t^{-3} + 2t^{-1}\right) = -16t^{-5} + 9t^{-4} - 2t^{-2} = -\dfrac{16}{t^5} + \dfrac{9}{t^4} - \dfrac{2}{t^2}$.

30. $f'(x) = \frac{d}{dx}\left(5x^{-3} - 2x^{-2} - x^{-1} + 200\right) = -15x^{-4} + 4x^{-3} + x^{-2} = -\dfrac{15}{x^4} + \dfrac{4}{x^3} + \dfrac{1}{x^2}$.

31. $f'(x) = \frac{d}{dx}\left(3x - 5x^{1/2}\right) = 3 - \frac{5}{2}x^{-1/2} = 3 - \dfrac{5}{2\sqrt{x}}$.

32. $f'(t) = \frac{d}{dt}\left(2t^2 + t^{3/2}\right) = 4t + \frac{3}{2}t^{1/2}$.

33. $f'(x) = \dfrac{d}{dx}\left(2x^{-2} - 3x^{-1/3}\right) = -4x^{-3} + x^{-4/3} = -\dfrac{4}{x^3} + \dfrac{1}{x^{4/3}}$.

34. $f'(x) = \dfrac{d}{dx}\left(\dfrac{3}{x^3} + \dfrac{4}{\sqrt{x}} + 1\right) = \dfrac{d}{dx}\left(3x^{-3} + 4x^{-1/2} + 1\right) = -9x^{-4} - 2x^{-3/2} = -\dfrac{9}{x^4} - \dfrac{2}{x^{3/2}}$.

35. $f'(x) = \frac{d}{dx}\left(2x^3 - 4x\right) = 6x^2 - 4$.

a. $f'(-2) = 6(-2)^2 - 4 = 20$. **b.** $f'(0) = 6(0) - 4 = -4$. **c.** $f'(2) = 6(2)^2 - 4 = 20$.

36. $f'(x) = \frac{d}{dx}\left(4x^{5/4} + 2x^{3/2} + x\right) = 5x^{1/4} + 3x^{1/2} + 1$.

a. $f'(4) = 5(4)^{1/4} + 3(4)^{1/2} + 1 = 5(4)^{1/4} + 6 + 1 = 5(4)^{1/4} + 7 = 5\sqrt{2} + 7$.

b. $f'(16) = 5(16)^{1/4} + 3(16)^{1/2} + 1 = 10 + 12 + 1 = 23$.

37. The given limit is $f'(1)$, where $f(x) = x^3$. Because $f'(x) = 3x^2$, we have $\lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = f'(1) = 3$.

38. Letting $h = x - 1$ or $x = h + 1$ and observing that $h \to 0$ as $x \to 1$, we find $\lim_{x\to 1} \frac{x^5 - 1}{x - 1} = \lim_{h\to 0} \frac{(h+1)^5 - 1}{h} = f'(1)$, where $f(x) = x^5$. Because $f'(x) = 5x^4$, we have $f'(1) = 5$, the value of the limit; that is, $\lim_{x\to 1} \frac{x^5 - 1}{x - 1} = 5$.

39. Let $f(x) = 3x^2 - x$. Then $\lim_{h\to 0} \frac{3(2+h)^2 - (2+h) - 10}{h} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$ because $f(2+h) - f(2) = 3(2+h)^2 - (2+h) - [3(4) - 2] = 3(2+h)^2 - (2+h) - 10$. But the last limit is simply $f'(2)$. Because $f'(x) = 6x - 1$, we have $f'(2) = 11$. Therefore, $\lim_{h\to 0} \frac{3(2+h)^2 - (2+h) - 10}{h} = 11$.

40. Write $\lim_{t\to 0} \frac{1 - (1+t)^2}{t(1+t)^2} = \lim_{t\to 0} \frac{1}{(1+t)^2} \cdot \lim_{t\to 0} \frac{1 - (1+t)^2}{t}$. Now let $f(t) = -t^2$. Then $\lim_{t\to 0} \frac{1 - (1+t)^2}{t} = \lim_{t\to 0} \frac{f(1+t) - f(1)}{t} = f'(1)$. Because $f'(t) = -2t$, we find $f'(1) = -2$. Therefore, $\lim_{t\to 0} \frac{1 - (1+t)^2}{t(1+t)^2} = \lim_{t\to 0} \frac{1}{(1+t)^2} \cdot f'(1) = 1 \cdot (-2) = -2$.

41. $f(x) = 2x^2 - 3x + 4$. The slope of the tangent line at any point $(x, f(x))$ on the graph of f is $f'(x) = 4x - 3$. In particular, the slope of the tangent line at the point $(2, 6)$ is $f'(2) = 4(2) - 3 = 5$. An equation of the required tangent line is $y - 6 = 5(x - 2)$ or $y = 5x - 4$.

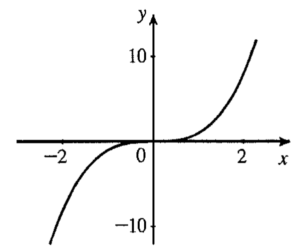
42. $f(x) = -\frac{5}{3}x^2 + 2x + 2$, so $f'(x) = -\frac{10}{3}x + 2$. The slope is $f'(-1) = \frac{10}{3} + 2 = \frac{16}{3}$. An equation of the tangent line is $y + \frac{5}{3} = \frac{16}{3}(x + 1)$ or $y = \frac{16}{3}x + \frac{11}{3}$.

43. $f(x) = x^4 - 3x^3 + 2x^2 - x + 1$, so $f'(x) = 4x^3 - 9x^2 + 4x - 1$. The slope is $f'(2) = 4(2)^3 - 9(2)^2 + 4(2) - 1 = 3$. An equation of the tangent line is $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

44. $f(x) = \sqrt{x} + 1/\sqrt{x}$. The slope of the tangent line at any point $(x, f(x))$ on the graph of f is $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$. In particular, the slope of the tangent line at the point $\left(4, \frac{5}{2}\right)$ is $f'(4) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$. An equation of the required tangent line is $y - \frac{5}{2} = \frac{3}{16}(x - 4)$ or $y = \frac{3}{16}x + \frac{7}{4}$.

45. a. $f'(x) = 3x^2$. At a point where the tangent line is horizontal, $f'(x) = 0$, or $3x^2 = 0$, and so $x = 0$. Therefore, the point is $(0, 0)$.

b.

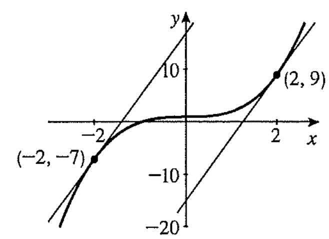


46. $f(x) = x^3 - 4x^2$, so $f'(x) = 3x^2 - 8x = x(3x - 8)$. Thus, $f'(x) = 0$ if $x = 0$ or $x = \frac{8}{3}$. Therefore, the points are $(0, 0)$ and $\left(\frac{8}{3}, -\frac{256}{27}\right)$.

47. a. $f(x) = x^3 + 1$. The slope of the tangent line at any point $(x, f(x))$ on the graph of f is $f'(x) = 3x^2$. At the point(s) where the slope is 12, we have $3x^2 = 12$, so $x = \pm 2$. The required points are $(-2, -7)$ and $(2, 9)$.

b. The tangent line at $(-2, -7)$ has equation $y - (-7) = 12[x - (-2)]$, or $y = 12x + 17$, and the tangent line at $(2, 9)$ has equation $y - 9 = 12(x - 2)$, or $y = 12x - 15$.

c.



48. $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 6$, so $f'(x) = 2x^2 + 2x - 12$.

a. $f'(x) = -12$ gives $2x^2 + 2x - 12 = -12$, $2x^2 + 2x = 0$, $2x(x + 1) = 0$; that is, $x = 0$ or $x = -1$.

b. $f'(x) = 0$ gives $2x^2 + 2x - 12 = 0$, $2(x^2 + x - 6) = 2(x + 3)(x - 2) = 0$, and so $x = -3$ or $x = 2$.

c. $f'(x) = 12$ gives $2x^2 + 2x - 12 = 12$, $2(x^2 + x - 12) = 2(x + 4)(x - 3) = 0$, and so $x = -4$ or $x = 3$.

49. $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2$, so $f'(x) = x^3 - x^2 - 2x$.

a. $f'(x) = x^3 - x^2 - 2x = -2x$ implies $x^3 - x^2 = 0$, so $x^2(x - 1) = 0$. Thus, $x = 0$ or $x = 1$. $f(1) = \frac{1}{4}(1)^4 - \frac{1}{3}(1)^3 - (1)^2 = -\frac{13}{12}$ and $f(0) = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - (0)^2 = 0$. We conclude that the corresponding points on the graph are $\left(1, -\frac{13}{12}\right)$ and $(0, 0)$.

b. $f'(x) = x^3 - x^2 - 2x = 0$ implies $x(x^2 - x - 2) = 0$, $x(x - 2)(x + 1) = 0$, and so $x = 0, 2$, or -1. $f(0) = 0$, $f(2) = \frac{1}{4}(2)^4 - \frac{1}{3}(2)^3 - (2)^2 = 4 - \frac{8}{3} - 4 = -\frac{8}{3}$, and $f(-1) = \frac{1}{4}(-1)^4 - \frac{1}{3}(-1)^3 - (-1)^2 = \frac{1}{4} + \frac{1}{3} - 1 = -\frac{5}{12}$. We conclude that the corresponding points are $(0, 0)$, $\left(2, -\frac{8}{3}\right)$, and $\left(-1, -\frac{5}{12}\right)$.

c. $f'(x) = x^3 - x^2 - 2x = 10x$ implies $x^3 - x^2 - 12x = 0$, $x(x^2 - x - 12) = 0$, $x(x - 4)(x + 3) = 0$, so $x = 0, 4$, or -3. $f(0) = 0$, $f(4) = \frac{1}{4}(4)^4 - \frac{1}{3}(4)^3 - (4)^2 = 48 - \frac{64}{3} = \frac{80}{3}$, and $f(-3) = \frac{1}{4}(-3)^4 - \frac{1}{3}(-3)^3 - (-3)^2 = \frac{81}{4} + 9 - 9 = \frac{81}{4}$. We conclude that the corresponding points are $(0, 0)$, $\left(4, \frac{80}{3}\right)$, and $\left(-3, \frac{81}{4}\right)$.

74. a. At any time t, the function $D = g + f$ at t, $D(t) = (g + f)(t) = g(t) + f(t)$, gives the total population aged 65 and over of the developed and the underdeveloped/emerging countries.

b. $D(t) = g(t) + f(t) = (0.46t^2 + 0.16t + 287.8) + (3.567t + 175.2) = 0.46t^2 + 3.727t + 463$, so $D'(t) = 0.92t + 3.727$. Therefore, $D'(10) = 0.92(10) + 3.727 = 12.927$, which says that the combined population is growing at the rate of approximately 13 million people per year in 2010.

75. a. $$G(t) = J(t) - N(t) = \begin{cases} -0.0002t^2 + 0.032t + 0.1 & \text{if } 0 \le t < 5 \\ 0.0002t^2 - 0.006t + 0.28 & \text{if } 5 \le t < 10 \\ -0.0012t^2 + 0.082t - 0.46 & \text{if } 10 \le t < 15 \end{cases}$$

b. In 2008, where $t = 8$, the gap is changing at a rate of $G'(8) = \left[\frac{d}{dt}\left(0.0002t^2 - 0.006t + 0.28\right)\right]_{t=8} = (0.0004t - 0.006)|_{t=8} = -0.0028$; that is, the gap is narrowing at a rate of 2800 jobs/yr. In 2012, where $t = 12$, the gap is changing at a rate of $G'(12) = \left[\frac{d}{dt}\left(-0.0012t^2 + 0.082t - 0.46\right)\right]_{t=12} = (-0.0024t + 0.082)|_{t=12} = 0.0532$; that is, the gap is increasing at a rate of 53,200 jobs/yr.

76. True. $\frac{d}{dx}[2f(x) - 5g(x)] = \frac{d}{dx}[2f(x)] - \frac{d}{dx}[5g(x)] = 2f'(x) - 5g'(x)$.

77. False. f is not a power function.

78. $$\frac{d}{dx}\left(x^3\right) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0}\frac{h\left(3x^2 + 3xh + h^2\right)}{h}$$
$$= \lim_{h\to 0}\left(3x^2 + 3xh + h^2\right) = 3x^2.$$

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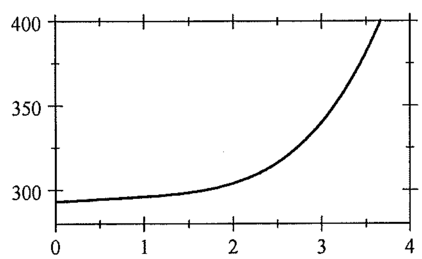
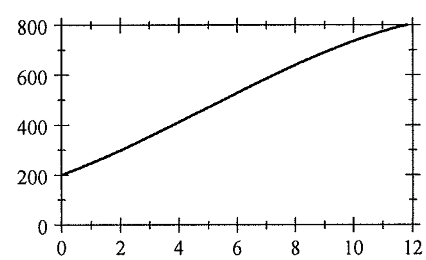
1. 1 **2.** 3.0720 **3.** 0.4226 **4.** 0.0732 **5.** 0.1613 **6.** 3.9730

7. a.



b. 3.4295 parts/million per 40 years;
164.239 parts/million per 40 years

8. a.



b. 42,272 cases/year