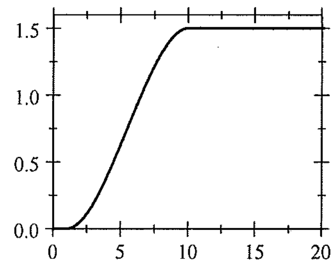
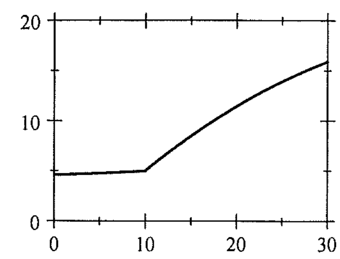
11.



12.



2.6 The Derivative

Concept Questions

page 148

1. a. $m = \dfrac{f(2+h) - f(2)}{h}$

b. The slope of the tangent line is $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$.

2. a. The average rate of change is $\dfrac{f(2+h) - f(2)}{h}$.

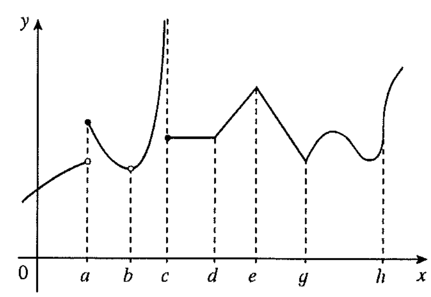
b. The instantaneous rate of change of f at 2 is $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$.

c. The expression for the slope of the secant line is the same as that for the average rate of change. The expression for the slope of the tangent line is the same as that for the instantaneous rate of change.

3. a. The expression $\dfrac{f(x+h) - f(x)}{h}$ gives (i) the slope of the secant line passing through the points $(x, f(x))$ and $(x+h, f(x+h))$, and (ii) the average rate of change of f over the interval $[x, x+h]$.

b. The expression $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ gives (i) the slope of the tangent line to the graph of f at the point $(x, f(x))$, and (ii) the instantaneous rate of change of f at x.

4. Loosely speaking, a function f does not have a derivative at a if the graph of f does not have a tangent line at a, or if the tangent line does exist, but is vertical. In the figure, the function fails to be differentiable at $x = a$, b, and c because it is discontinuous at each of these numbers. The derivative of the function does not exist at $x = d$, e, and g because it has a kink at each point on the graph corresponding to these numbers. Finally, the function is not differentiable at $x = h$ because the tangent line is vertical at $(h, f(h))$.



5. a. $C(500)$ gives the total cost incurred in producing 500 units of the product.

b. $C'(500)$ gives the rate of change of the total cost function when the production level is 500 units.

6. a. $P(5)$ gives the population of the city (in thousands) when $t = 5$.

b. $P'(5)$ gives the rate of change of the city's population (in thousands/year) when $t = 5$.

Exercises page 149

1. The rate of change of the average infant's weight when $t = 3$ is $\frac{7.5}{5}$, or 1.5 lb/month. The rate of change of the average infant's weight when $t = 18$ is $\frac{3.5}{6}$, or approximately 0.58 lb/month. The average rate of change over the infant's first year of life is $\frac{22.5-7.5}{12}$, or 1.25 lb/month.

2. The rate at which the wood grown is changing at the beginning of the 10th year is $\frac{4}{12}$, or $\frac{1}{3}$ cubic meter per hectare per year. At the beginning of the 30th year, it is $\frac{10}{8}$, or 1.25 cubic meters per hectare per year.

3. The rate of change of the percentage of households watching television at 4 p.m. is $\frac{12.3}{4}$, or approximately 3.1 percent per hour. The rate at 11 p.m. is $\frac{-42.3}{2} = -21.15$, that is, it is dropping off at the rate of 21.15 percent per hour.

4. The rate of change of the crop yield when the density is 200 aphids per bean stem is $\frac{-500}{300}$, a decrease of approximately 1.7 kg/4000 m^2 per aphid per bean stem. The rate of change when the density is 800 aphids per bean stem is $\frac{-150}{300}$, a decrease of approximately 0.5 kg/4000 m^2 per aphid per bean stem.

5. **a.** Car A is travelling faster than Car B at t_1 because the slope of the tangent line to the graph of f is greater than the slope of the tangent line to the graph of g at t_1.

   b. Their speed is the same because the slope of the tangent lines are the same at t_2.

   c. Car B is travelling faster than Car A.

   d. They have both covered the same distance and are once again side by side at t_3.

6. **a.** At t_1, the velocity of Car A is greater than that of Car B because $f(t_1) > g(t_1)$. However, Car B has greater acceleration because the slope of the tangent line to the graph of g is increasing, whereas the slope of the tangent line to f is decreasing as you move across t_1.

   b. Both cars have the same velocity at t_2, but the acceleration of Car B is greater than that of Car A because the slope of the tangent line to the graph of g is increasing, whereas the slope of the tangent line to the graph of f is decreasing as you move across t_2.

7. **a.** P_2 is decreasing faster at t_1 because the slope of the tangent line to the graph of g at t_1 is greater than the slope of the tangent line to the graph of f at t_1.

   b. P_1 is decreasing faster than P_2 at t_2.

   c. Bactericide B is more effective in the short run, but bactericide A is more effective in the long run.

8. **a.** The revenue of the established department store is decreasing at the slowest rate at $t = 0$.

   b. The revenue of the established department store is decreasing at the fastest rate at t_3.

   c. The revenue of the discount store first overtakes that of the established store at t_1.

   d. The revenue of the discount store is increasing at the fastest rate at t_2 because the slope of the tangent line to the graph of f is greatest at the point $(t_2, f(t_2))$.

9. $f(x) = 13.$

Step 1 $f(x+h) = 13.$

Step 2 $f(x+h) - f(x) = 13 - 13 = 0.$

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{0}{h} = 0.$

Step 4 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} 0 = 0.$

10. $f(x) = -6.$

Step 1 $f(x+h) = -6.$

Step 2 $f(x+h) - f(x) = -6 - (-6) = 0.$

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{0}{h} = 0.$

Step 4 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} 0 = 0.$

11. $f(x) = 2x + 7.$

Step 1 $f(x+h) = 2(x+h) + 7.$

Step 2 $f(x+h) - f(x) = 2(x+h) + 7 - (2x+7) = 2h.$

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2h}{h} = 2.$

Step 4 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} 2 = 2.$

12. $f(x) = 8 - 4x.$

Step 1 $f(x+h) = 8 - 4(x+h) = 8 - 4x - 4h.$

Step 2 $f(x+h) - f(x) = (8 - 4x - 4h) - (8 - 4x) = -4h.$

Step 3 $\dfrac{f(x+h) - f(x)}{h} = -\dfrac{4h}{h} = -4.$

Step 4 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} (-4) = -4.$

13. $f(x) = 3x^2.$

Step 1 $f(x+h) = 3(x+h)^2 = 3x^2 + 6xh + 3h^2.$

Step 2 $f(x+h) - f(x) = \left(3x^2 + 6xh + 3h^2\right) - 3x^2 = 6xh + 3h^2 = h(6x + 3h).$

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(6x+3h)}{h} = 6x + 3h.$

Step 4 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} (6x + 3h) = 6x.$

14. $f(x) = -\frac{1}{2}x^2.$

Step 1 $f(x+h) = -\frac{1}{2}(x+h)^2.$

Step 2 $f(x+h) - f(x) = -\frac{1}{2}x^2 - xh - \frac{1}{2}h^2 + \frac{1}{2}x^2 = -h\left(x + \frac{1}{2}h\right).$

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-h\left(x + \frac{1}{2}h\right)}{h} = -\left(x + \frac{1}{2}h\right).$

Step 4 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} -\left(x + \frac{1}{2}h\right) = -x.$

15. $f(x) = -x^2 + 3x$.

Step 1 $f(x+h) = -(x+h)^2 + 3(x+h) = -x^2 - 2xh - h^2 + 3x + 3h$.

Step 2 $f(x+h) - f(x) = \left(-x^2 - 2xh - h^2 + 3x + 3h\right) - \left(-x^2 + 3x\right) = -2xh - h^2 + 3h$

$= h(-2x - h + 3)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(-2x - h + 3)}{h} = -2x - h + 3$.

Step 4 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} (-2x - h + 3) = -2x + 3$.

16. $f(x) = 2x^2 + 5x$.

Step 1 $f(x+h) = 2(x+h)^2 + 5(x+h) = 2x^2 + 4xh + 2h^2 + 5x + 5h$.

Step 2 $f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 + 5x + 5h - 2x^2 - 5x = h(4x + 2h + 5)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(4x + 2h + 5)}{h} = 4x + 2h + 5$.

Step 4 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} (4x + 2h + 5) = 4x + 5$.

17. $f(x) = 2x + 7$.

Step 1 $f(x+h) = 2(x+h) + 7 = 2x + 2h + 7$.

Step 2 $f(x+h) - f(x) = 2x + 2h + 7 - 2x - 7 = 2h$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{2h}{h} = 2$.

Step 4 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} 2 = 2$.

Therefore, $f'(x) = 2$. In particular, the slope at $x = 2$ is 2. Therefore, an equation of the tangent line is $y - 11 = 2(x - 2)$ or $y = 2x + 7$.

18. $f(x) = -3x + 4$. First, we find $f'(x) = -3$ using the four-step process. Thus, the slope of the tangent line is $f'(-1) = -3$ and an equation is $y - 7 = -3(x + 1)$ or $y = -3x + 4$.

19. $f(x) = 3x^2$.We first compute $f'(x) = 6x$ (see Exercise 13). Because the slope of the tangent line is $f'(1) = 6$, we use the point-slope form of the equation of a line and find that an equation is $y - 3 = 6(x - 1)$, or $y = 6x - 3$.

20. $f(x) = 3x - x^2$.

Step 1 $f(x+h) = 3(x+h) - (x+h)^2 = 3x + 3h - x^2 - 2xh - h^2$.

Step 2 $f(x+h) - f(x) = 3x + 3h - x^2 - 2xh - h^2 - 3x + x^2 = 3h - 2xh - h^2 = h(3 - 2x - h)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(3 - 2x - h)}{h} = 3 - 2x - h$.

Step 4 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} (3 - 2x - h) = 3 - 2x$.

Therefore, $f'(x) = 3 - 2x$. In particular, $f'(-2) = 3 - 2(-2) = 7$. Using the point-slope form of an equation of a line, we find $y + 10 = 7(x + 2)$, or $y = 7x + 4$.

21. $f(x) = -1/x$. We first compute $f'(x)$ using the four-step process:

Step 1 $f(x+h) = -\dfrac{1}{x+h}$.

Step 2 $f(x+h) - f(x) = -\dfrac{1}{x+h} + \dfrac{1}{x} = \dfrac{-x + (x+h)}{x(x+h)} = \dfrac{h}{x(x+h)}$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\frac{h}{x(x+h)}}{h} = \dfrac{1}{x(x+h)}$.

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{1}{x(x+h)} = \dfrac{1}{x^2}$.

The slope of the tangent line is $f'(3) = \frac{1}{9}$. Therefore, an equation is $y - \left(-\frac{1}{3}\right) = \frac{1}{9}(x-3)$, or $y = \frac{1}{9}x - \frac{2}{3}$.

22. $f(x) = \dfrac{3}{2x}$. First use the four-step process to find $f'(x) = -\dfrac{3}{2x^2}$. (This is similar to Exercise 21.) The slope of the tangent line is $f'(1) = -\frac{3}{2}$. Therefore, an equation is $y - \frac{3}{2} = -\frac{3}{2}(x-1)$ or $y = -\frac{3}{2}x + 3$.

23. **a.** $f(x) = 2x^2 + 1$.

Step 1 $f(x+h) = 2(x+h)^2 + 1 = 2x^2 + 4xh + 2h^2 + 1$.

Step 2 $f(x+h) - f(x) = \left(2x^2 + 4xh + 2h^2 + 1\right) - \left(2x^2 + 1\right)$

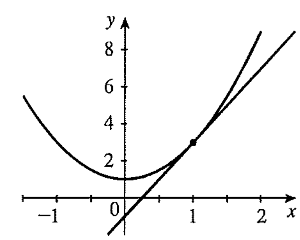
$= 4xh + 2h^2 = h(4x + 2h)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(4x+2h)}{h} = 4x + 2h$.

Step 4 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (4x + 2h) = 4x$.

b. The slope of the tangent line is $f'(1) = 4(1) = 4$. Therefore, an equation is $y - 3 = 4(x-1)$ or $y = 4x - 1$.

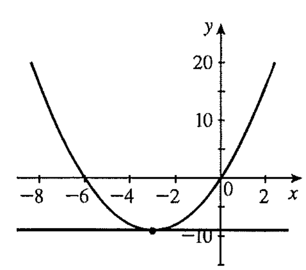
c.



24. **a.** $f(x) = x^2 + 6x$. Using the four-step process, we find that $f'(x) = 2x + 6$.

b. At a point on the graph of f where the tangent line to the curve is horizontal, $f'(x) = 0$. Then $2x + 6 = 0$, or $x = -3$. Therefore, $y = f(-3) = (-3)^2 + 6(-3) = -9$. The required point is $(-3, -9)$.

c.



25. **a.** $f(x) = x^2 - 2x + 1$. We use the four-step process:

Step 1 $f(x+h) = (x+h)^2 - 2(x+h) + 1 = x^2 + 2xh + h^2 - 2x - 2h + 1$.

Step 2 $f(x+h) - f(x) = \left(x^2 + 2xh + h^2 - 2x - 2h + 1\right) - \left(x^2 - 2x + 1\right) = 2xh + h^2 - 2h$

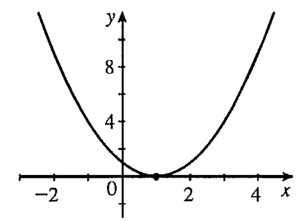
$= h(2x + h - 2)$.

Step 3 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{h(2x+h-2)}{h} = 2x+h-2.$

Step 4 $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2x+h-2)$
$= 2x-2.$

b. At a point on the graph of f where the tangent line to the curve is horizontal, $f'(x)=0$. Then $2x-2=0$, or $x=1$. Because $f(1)=1-2+1=0$, we see that the required point is $(1,0)$.

c.



d. It is changing at the rate of 0 units per unit change in x.

26. a. $f(x) = \dfrac{1}{x-1}.$

Step 1 $f(x+h) = \dfrac{1}{(x+h)-1} = \dfrac{1}{x+h-1}.$

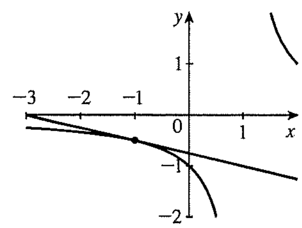
Step 2 $f(x+h)-f(x) = \dfrac{1}{x+h-1} - \dfrac{1}{x-1} = \dfrac{x-1-(x+h-1)}{(x+h-1)(x-1)} = -\dfrac{h}{(x+h-1)(x-1)}.$

Step 3 $\dfrac{f(x+h)-f(x)}{h} = -\dfrac{1}{(x+h-1)(x-1)}.$

Step 4 $f'(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0} -\dfrac{1}{(x+h-1)(x-1)} = -\dfrac{1}{(x-1)^2}.$

b. The slope is $f'(-1) = -\frac{1}{4}$, so, an equation is
$y - \left(-\frac{1}{2}\right) = -\frac{1}{4}(x+1)$ or $y = -\frac{1}{4}x - \frac{3}{4}$.

c.



27. a. $f(x) = x^2 + x$, so $\dfrac{f(3)-f(2)}{3-2} = \dfrac{(3^2+3)-(2^2+2)}{1} = 6,$

$\dfrac{f(2.5)-f(2)}{2.5-2} = \dfrac{(2.5^2+2.5)-(2^2+2)}{0.5} = 5.5$, and $\dfrac{f(2.1)-f(2)}{2.1-2} = \dfrac{(2.1^2+2.1)-(2^2+2)}{0.1} = 5.1.$

b. We first compute $f'(x)$ using the four-step process.

Step 1 $f(x+h) = (x+h)^2 + (x+h) = x^2+2xh+h^2+x+h.$

Step 2 $f(x+h)-f(x) = \left(x^2+2xh+h^2+x+h\right) - \left(x^2+x\right) = 2xh+h^2+h = h(2x+h+1).$

Step 3 $\dfrac{f(x+h)-f(x)}{h} = \dfrac{h(2x+h+1)}{h} = 2x+h+1.$

Step 4 $f'(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2x+h+1) = 2x+1.$

The instantaneous rate of change of y at $x=2$ is $f'(2) = 2(2)+1$, or 5 units per unit change in x.

c. The results of part (a) suggest that the average rates of change of f at $x=2$ approach 5 as the interval $[2, 2+h]$ gets smaller and smaller ($h = 1$, 0.5, and 0.1). This number is the instantaneous rate of change of f at $x=2$ as computed in part (b).

28. a. $f(x) = x^2 - 4x$, so $\dfrac{f(4)-f(3)}{4-3} = \dfrac{(16-16)-(9-12)}{1} = 3,$

$\dfrac{f(3.5)-f(3)}{3.5-3} = \dfrac{(12.25-14)-(9-12)}{0.5} = 2.5$, and $\dfrac{f(3.1)-f(3)}{3.1-3} = \dfrac{(9.61-12.4)-(9-12)}{0.1} = 2.1.$

b. We first compute $f'(x)$ using the four-step process:

Step 1 $f(x+h) = (x+h)^2 - 4(x+h) = x^2 + 2xh + h^2 - 4x - 4h$.

Step 2 $f(x+h) - f(x) = (x^2 + 2xh + h^2 - 4x - 4h) - (x^2 - 4x) = 2xh + h^2 - 4h = h(2x + h - 4)$.

Step 3 $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(2x+h-4)}{h} = 2x + h - 4$.

Step 4 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} (2x + h - 4) = 2x - 4$.

The instantaneous rate of change of y at $x = 3$ is $f'(3) = 6 - 4 = 2$, or 2 units per unit change in x.

c. The results of part (a) suggest that the average rates of change of f over smaller and smaller intervals containing $x = 3$ approach the instantaneous rate of change of 2 units per unit change in x obtained in part (b).

29. a. $f(t) = 2t^2 + 48t$. The average velocity of the car over the time interval $[20, 21]$ is

$\dfrac{f(21) - f(20)}{21 - 20} = \dfrac{[2(21)^2 + 48(21)] - [2(20)^2 + 48(20)]}{1} = 130\ \dfrac{\text{ft}}{\text{s}}$. Its average velocity over $[20, 20.1]$ is

$\dfrac{f(20.1) - f(20)}{20.1 - 20} = \dfrac{[2(20.1)^2 + 48(20.1)] - [2(20)^2 + 48(20)]}{0.1} = 128.2\ \dfrac{\text{ft}}{\text{s}}$. Its average velocity over

$[20, 20.01]$ is $\dfrac{f(20.01) - f(20)}{20.01 - 20} = \dfrac{[2(20.01)^2 + 48(20.01)] - [2(20)^2 + 48(20)]}{0.01} = 128.02\ \dfrac{\text{ft}}{\text{s}}$.

b. We first compute $f'(t)$ using the four-step process.

Step 1 $f(t+h) = 2(t+h)^2 + 48(t+h) = 2t^2 + 4th + 2h^2 + 48t + 48h$.

Step 2 $f(t+h) - f(t) = (2t^2 + 4th + 2h^2 + 48t + 48h) - (2t^2 + 48t) = 4th + 2h^2 + 48h$

$= h(4t + 2h + 48)$.

Step 3 $\dfrac{f(t+h) - f(t)}{h} = \dfrac{h(4t + 2h + 48)}{h} = 4t + 2h + 48$.

Step 4 $f'(t) = \lim\limits_{t\to 0} \dfrac{f(t+h) - f(t)}{h} = \lim\limits_{t\to 0} (4t + 2h + 48) = 4t + 48$.

The instantaneous velocity of the car at $t = 20$ is $f'(20) = 4(20) + 48$, or 128 ft/s.

c. Our results show that the average velocities do approach the instantaneous velocity as the intervals over which they are computed decreases.

30. a. The average velocity of the ball over the time interval $[2, 3]$ is

$\dfrac{s(3) - s(2)}{3 - 2} = \dfrac{[128(3) - 16(3)^2] - [128(2) - 16(2)^2]}{1} = 48$, or 48 ft/s. Over the time interval $[2, 2.5]$, it is

$\dfrac{s(2.5) - s(2)}{2.5 - 2} = \dfrac{[128(2.5) - 16(2.5)^2] - [128(2) - 16(2)^2]}{0.5} = 56$, or 56 ft/s. Over the time interval $[2, 2.1]$,

it is $\dfrac{s(2.1) - s(2)}{2.1 - 2} = \dfrac{[128(2.1) - 16(2.1)^2] - [128(2) - 16(2)^2]}{0.1} = 62.4$, or 62.4 ft/s.

b. Using the four-step process, we find that the instantaneous velocity of the ball at any time t is given by $v(t) = 128 - 32t$. In particular, the velocity of the ball at $t = 2$ is $v(2) = 128 - 32(2) = 64$, or 64 ft/s.

c. At $t = 5$, $v(5) = 128 - 32(5) = -32$, so the speed of the ball at $t = 5$ is 32 ft/s and it is falling.

d. The ball hits the ground when $s(t) = 0$, that is, when $128t - 16t^2 = 0$, whence $t(128 - 16t) = 0$, so $t = 0$ or $t = 8$. Thus, it will hit the ground when $t = 8$.

31. a. We solve the equation $16t^2 = 400$ and find $t = 5$, which is the time it takes the screwdriver to reach the ground.