2.2 The Algebra of Functions

Concept Questions page 73

1. a. $P(x_1) = R(x_1) - C(x_1)$ gives the profit if x_1 units are sold.

   b. $P(x_2) = R(x_2) - C(x_2)$. Because $P(x_2) < 0$, $|R(x_2) - C(x_2)| = -[R(x_2) - C(x_2)]$ gives the loss sustained if x_2 units are sold.

2. a. $(f+g)(x) = f(x) + g(x)$, $(f-g)(x) = f(x) - g(x)$, and $(fg)(x) = f(x)g(x)$; all have domain $A \cap B$. $(f/g)(x) = \dfrac{f(x)}{g(x)}$ has domain $A \cap B$ excluding $x \in A \cap B$ such that $g(x) = 0$.

   b. $(f+g)(2) = f(2) + g(2) = 3 + (-2) = 1$, $(f-g)(2) = f(2) - g(2) = 3 - (-2) = 5$, $(fg)(2) = f(2)g(2) = 3(-2) = -6$, and $(f/g)(2) = \dfrac{f(2)}{g(2)} = \dfrac{3}{-2} = -\dfrac{3}{2}$

3. a. $y = (f+g)(x) = f(x) + g(x)$

   b. $y = (f-g)(x) = f(x) - g(x)$

   c. $y = (fg)(x) = f(x)g(x)$

   d. $y = \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

4. a. The domain of $(f \circ g)(x) = f(g(x))$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. The domain of $(g \circ f)(x) = g(f(x))$ is the set of all x in the domain of f such that $f(x)$ is in the domain of g.

   b. $(g \circ f)(2) = g(f(2)) = g(3) = 8$. We cannot calculate $(f \circ g)(3)$ because $(f \circ g)(3) = f(g(3)) = f(8)$, and we don't know the value of $f(8)$.

5. No. Let $A = (-\infty, \infty)$, $f(x) = x$, and $g(x) = \sqrt{x}$. Then $a = -1$ is in A, but $(g \circ f)(-1) = g(f(-1)) = g(-1) = \sqrt{-1}$ is not defined.

6. The required expression is $P = g(f(p))$.

Exercises page 74

1. $(f+g)(x) = f(x) + g(x) = (x^3+5) + (x^2-2) = x^3 + x^2 + 3$.

2. $(f-g)(x) = f(x) - g(x) = (x^3+5) - (x^2-2) = x^3 - x^2 + 7$.

3. $fg(x) = f(x)g(x) = (x^3+5)(x^2-2) = x^5 - 2x^3 + 5x^2 - 10$.

4. $gf(x) = g(x)f(x) = (x^2-2)(x^3+5) = x^5 - 2x^3 + 5x^2 - 10$.

5. $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \dfrac{x^3+5}{x^2-2}$.

6. $\dfrac{f-g}{h}(x) = \dfrac{f(x)-g(x)}{h(x)} = \dfrac{x^3+5-(x^2-2)}{2x+4} = \dfrac{x^3-x^2+7}{2x+4}$.

7. $\dfrac{fg}{h}(x) = \dfrac{f(x)g(x)}{h(x)} = \dfrac{(x^3+5)(x^2-2)}{2x+4} = \dfrac{x^5-2x^3+5x^2-10}{2x+4}$.

8. $fgh(x) = f(x)g(x)h(x) = (x^3+5)(x^2-2)(2x+4) = (x^5-2x^3+5x^2-10)(2x+4)$
$= 2x^6 - 4x^4 + 10x^3 - 20x + 4x^5 - 8x^3 + 20x^2 - 40 = 2x^6 + 4x^5 - 4x^4 + 2x^3 + 20x^2 - 20x - 40.$

9. $(f+g)(x) = f(x) + g(x) = x - 1 + \sqrt{x+1}.$

10. $(g-f)(x) = g(x) - f(x) = \sqrt{x+1} - (x-1) = \sqrt{x+1} - x + 1.$

11. $(fg)(x) = f(x)g(x) = (x-1)\sqrt{x+1}.$

12. $(gf)(x) = g(x)f(x) = \sqrt{x+1}(x-1).$

13. $\frac{g}{h}(x) = \frac{g(x)}{h(x)} = \frac{\sqrt{x+1}}{2x^3-1}.$

14. $\frac{h}{g}(x) = \frac{h(x)}{g(x)} = \frac{2x^3-1}{\sqrt{x+1}}.$

15. $\frac{fg}{h}(x) = \frac{(x-1)(\sqrt{x+1})}{2x^3-1}.$

16. $\frac{fh}{g}(x) = \frac{(x-1)(2x^3-1)}{\sqrt{x+1}} = \frac{2x^4-2x^3-x+1}{\sqrt{x+1}}.$

17. $\frac{f-h}{g}(x) = \frac{x-1-(2x^3-1)}{\sqrt{x+1}} = \frac{x-2x^3}{\sqrt{x+1}}.$

18. $\frac{gh}{g-f}(x) = \frac{\sqrt{x+1}(2x^3-1)}{\sqrt{x+1}-(x-1)} = \frac{\sqrt{x+1}(2x^3-1)}{\sqrt{x+1}-x+1}.$

19. $(f+g)(x) = x^2 + 5 + \sqrt{x} - 2 = x^2 + \sqrt{x} + 3$, $(f-g)(x) = x^2 + 5 - (\sqrt{x} - 2) = x^2 - \sqrt{x} + 7$,
$(fg)(x) = (x^2+5)(\sqrt{x}-2)$, and $\left(\frac{f}{g}\right)(x) = \frac{x^2+5}{\sqrt{x}-2}.$

20. $(f+g)(x) = \sqrt{x-1} + x^3 + 1$, $(f-g)(x) = \sqrt{x-1} - x^3 - 1$, $(fg)(x) = \sqrt{x-1}(x^3+1)$, and
$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x-1}}{x^3+1}.$

21. $(f+g)(x) = \sqrt{x+3} + \frac{1}{x-1} = \frac{(x-1)\sqrt{x+3}+1}{x-1}$, $(f-g)(x) = \sqrt{x+3} - \frac{1}{x-1} = \frac{(x-1)\sqrt{x+3}-1}{x-1}$,
$(fg)(x) = \sqrt{x+3}\left(\frac{1}{x-1}\right) = \frac{\sqrt{x+3}}{x-1}$, and $\left(\frac{f}{g}\right) = \sqrt{x+3}(x-1).$

22. $(f+g)(x) = \frac{1}{x^2+1} + \frac{1}{x^2-1} = \frac{x^2-1+x^2+1}{(x^2+1)(x^2-1)} = \frac{2x^2}{(x^2+1)(x^2-1)}$,
$(f-g)(x) = \frac{1}{x^2+1} - \frac{1}{x^2-1} = \frac{x^2-1-x^2-1}{(x^2+1)(x^2-1)} = -\frac{2}{(x^2+1)(x^2-1)}$, $(fg)(x) = \frac{1}{(x^2+1)(x^2-1)}$, and
$\left(\frac{f}{g}\right)(x) = \frac{x^2-1}{x^2+1}.$

23. $(f+g)(x) = \dfrac{x+1}{x-1} + \dfrac{x+2}{x-2} = \dfrac{(x+1)(x-2)+(x+2)(x-1)}{(x-1)(x-2)} = \dfrac{x^2-x-2+x^2+x-2}{(x-1)(x-2)}$

$= \dfrac{2x^2-4}{(x-1)(x-2)} = \dfrac{2(x^2-2)}{(x-1)(x-2)}$,

$(f-g)(x) = \dfrac{x+1}{x-1} - \dfrac{x+2}{x-2} = \dfrac{(x+1)(x-2)-(x+2)(x-1)}{(x-1)(x-2)} = \dfrac{x^2-x-2-x^2-x+2}{(x-1)(x-2)}$

$= \dfrac{-2x}{(x-1)(x-2)}$,

$(fg)(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$, and $\left(\dfrac{f}{g}\right)(x) = \dfrac{(x+1)(x-2)}{(x-1)(x+2)}$.

24. $(f+g)(x) = x^2+1+\sqrt{x+1}$, $(f-g)(x) = x^2+1-\sqrt{x+1}$, $(fg)(x) = (x^2+1)\sqrt{x+1}$, and $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2+1}{\sqrt{x+1}}$.

25. $(f \circ g)(x) = f(g(x)) = f(x^2) = (x^2)^2 + x^2 + 1 = x^4 + x^2 + 1$ and $(g \circ f)(x) = g(f(x)) = g(x^2+x+1) = (x^2+x+1)^2$.

26. $(f \circ g)(x) = f(g(x)) = 3[g(x)]^2 + 2g(x) + 1 = 3(x+3)^2 + 2(x+3) + 1 = 3x^2 + 20x + 34$ and $(g \circ f)(x) = g(f(x)) = f(x) + 3 = 3x^2 + 2x + 1 + 3 = 3x^2 + 2x + 4$.

27. $(f \circ g)(x) = f(g(x)) = f(x^2-1) = \sqrt{x^2-1} + 1$ and $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}+1) = (\sqrt{x}+1)^2 - 1 = x + 2\sqrt{x} + 1 - 1 = x + 2\sqrt{x}$.

28. $(f \circ g)(x) = f(g(x)) = 2\sqrt{g(x)} + 3 = 2\sqrt{x^2+1} + 3$ and $(g \circ f)(x) = g(f(x)) = [f(x)]^2 + 1 = (2\sqrt{x}+3)^2 + 1 = 4x + 12\sqrt{x} + 10$.

29. $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} \div \left(\dfrac{1}{x^2}+1\right) = \dfrac{1}{x} \cdot \dfrac{x^2}{x^2+1} = \dfrac{x}{x^2+1}$ and

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x}{x^2+1}\right) = \dfrac{x^2+1}{x}$.

30. $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x-1}\right) = \sqrt{\dfrac{x}{x-1}}$ and

$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x+1}\right) = \dfrac{1}{\sqrt{x+1}-1} \cdot \dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \dfrac{\sqrt{x+1}+1}{x}$.

31. $h(2) = g(f(2))$. But $f(2) = 2^2 + 2 + 1 = 7$, so $h(2) = g(7) = 49$.

32. $h(2) = g(f(2))$. But $f(2) = (2^2-1)^{1/3} = 3^{1/3}$, so $h(2) = g(3^{1/3}) = 3(3^{1/3})^3 + 1 = 3(3) + 1 = 10$.

33. $h(2) = g(f(2))$. But $f(2) = \dfrac{1}{2(2)+1} = \dfrac{1}{5}$, so $h(2) = g\left(\dfrac{1}{5}\right) = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$.

34. $h(2) = g(f(2))$. But $f(2) = \dfrac{1}{2-1} = 1$, so $g(1) = 1^2 + 1 = 2$.

35. $f(x) = 2x^3 + x^2 + 1, g(x) = x^5.$

36. $f(x) = 3x^2 - 4, g(x) = x^{-3}.$

37. $f(x) = x^2 - 1, g(x) = \sqrt{x}.$

38. $f(x) = (2x - 3), g(x) = x^{3/2}.$

39. $f(x) = x^2 - 1, g(x) = \frac{1}{x}.$

40. $f(x) = x^2 - 4, g(x) = \frac{1}{\sqrt{x}}.$

41. $f(x) = 3x^2 + 2, g(x) = \frac{1}{x^{3/2}}.$

42. $f(x) = \sqrt{2x + 1}, g(x) = \frac{1}{x} + x.$

43. $f(a + h) - f(a) = [3(a + h) + 4] - (3a + 4) = 3a + 3h + 4 - 3a - 4 = 3h.$

44. $f(a + h) - f(a) = -\frac{1}{2}(a + h) + 3 - \left(-\frac{1}{2}a + 3\right) = -\frac{1}{2}a - \frac{1}{2}h + 3 + \frac{1}{2}a - 3 = -\frac{1}{2}h.$

45. $f(a + h) - f(a) = 4 - (a + h)^2 - \left(4 - a^2\right) = 4 - a^2 - 2ah - h^2 - 4 + a^2 = -2ah - h^2 = -h(2a + h).$

46.
$$f(a + h) - f(a) = \left[(a + h)^2 - 2(a + h) + 1\right] - \left(a^2 - 2a + 1\right)$$
$$= a^2 + 2ah + h^2 - 2a - 2h + 1 - a^2 + 2a - 1 = h(2a + h - 2).$$

47.
$$\frac{f(a + h) - f(a)}{h} = \frac{\left[(a + h)^2 + 1\right] - \left(a^2 + 1\right)}{h} = \frac{a^2 + 2ah + h^2 + 1 - a^2 - 1}{h} = \frac{2ah + h^2}{h}$$
$$= \frac{h(2a + h)}{h} = 2a + h.$$

48.
$$\frac{f(a + h) - f(a)}{h} = \frac{\left[2(a + h)^2 - (a + h) + 1\right] - \left(2a^2 - a + 1\right)}{h}$$
$$= \frac{2a^2 + 4ah + 2h^2 - a - h + 1 - 2a^2 + a - 1}{h} = \frac{4ah + 2h^2 - h}{h} = 4a + 2h - 1.$$

49.
$$\frac{f(a + h) - f(a)}{h} = \frac{\left[(a + h)^3 - (a + h)\right] - \left(a^3 - a\right)}{h} = \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a - h - a^3 + a}{h}$$
$$= \frac{3a^2h + 3ah^2 + h^3 - h}{h} = 3a^2 + 3ah + h^2 - 1.$$

50.
$$\frac{f(a + h) - f(a)}{h} = \frac{\left[2(a + h)^3 - (a + h)^2 + 1\right] - \left(2a^3 - a^2 + 1\right)}{h}$$
$$= \frac{2a^3 + 6a^2h + 6ah^2 + 2h^3 - a^2 - 2ah - h^2 + 1 - 2a^3 + a^2 - 1}{h}$$
$$= \frac{6a^2h + 6ah^2 + 2h^3 - 2ah - h^2}{h} = 6a^2 + 6ah + 2h^2 - 2a - h.$$

51.
$$\frac{f(a + h) - f(a)}{h} = \frac{\frac{1}{a + h} - \frac{1}{a}}{h} = \frac{\frac{a - (a + h)}{a(a + h)}}{h} = -\frac{1}{a(a + h)}.$$

52.
$$\frac{f(a + h) - f(a)}{h} = \frac{\sqrt{a + h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a + h} + \sqrt{a}}{\sqrt{a + h} + \sqrt{a}} = \frac{(a + h) - a}{h\left(\sqrt{a + h} + \sqrt{a}\right)} = \frac{1}{\sqrt{a + h} + \sqrt{a}}.$$

53. $F(t)$ represents the total revenue for the two restaurants at time t.