2 FUNCTIONS, LIMITS, AND THE DERIVATIVE

2.1 Functions and Their Graphs

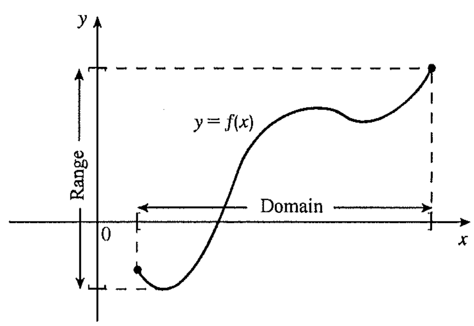
Concept Questions page 59

1. **a.** A function is a rule that associates with each element in a set A exactly one element in a set B.

   b. The domain of a function f is the set of all elements x in the set such that $f(x)$ is an element in B. The range of f is the set of all elements $f(x)$ whenever x is an element in its domain.

   c. An independent variable is a variable in the domain of a function f. The dependent variable is $y = f(x)$.

2. **a.** The graph of a function f is the set of all ordered pairs (x, y) such that $y = f(x)$, x being an element in the domain of f.



   b. Use the vertical line test to determine if every vertical line intersects the curve in at most one point. If so, then the curve is the graph of a function.

3. **a.** Yes, every vertical line intersects the curve in at most one point.

   b. No, a vertical line intersects the curve at more than one point.

   c. No, a vertical line intersects the curve at more than one point.

   d. Yes, every vertical line intersects the curve in at most one point.

4. The domain is $[1, 3) \cup [3, 5)$ and the range is $\left[\frac{1}{2}, 2\right) \cup (2, 4]$.

Exercises page 59

1. $f(x) = 5x + 6$. Therefore $f(3) = 5(3) + 6 = 21$, $f(-3) = 5(-3) + 6 = -9$, $f(a) = 5(a) + 6 = 5a + 6$, $f(-a) = 5(-a) + 6 = -5a + 6$, and $f(a + 3) = 5(a + 3) + 6 = 5a + 15 + 6 = 5a + 21$.

2. $f(x) = 4x - 3$. Therefore, $f(4) = 4(4) - 3 = 16 - 3 = 13$, $f\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right) - 3 = 1 - 3 = -2$, $f(0) = 4(0) - 3 = -3$, $f(a) = 4(a) - 3 = 4a - 3$, $f(a + 1) = 4(a + 1) - 3 = 4a + 1$.

3. $g(x) = 3x^2 - 6x - 3$, so $g(0) = 3(0) - 6(0) - 3 = -3$, $g(-1) = 3(-1)^2 - 6(-1) - 3 = 3 + 6 - 3 = 6$, $g(a) = 3(a)^2 - 6(a) - 3 = 3a^2 - 6a - 3$, $g(-a) = 3(-a)^2 - 6(-a) - 3 = 3a^2 + 6a - 3$, and $g(x+1) = 3(x+1)^2 - 6(x+1) - 3 = 3\left(x^2 + 2x + 1\right) - 6x - 6 - 3 = 3x^2 + 6x + 3 - 6x - 9 = 3x^2 - 6$.

4. $h(x) = x^3 - x^2 + x + 1$, so $h(-5) = (-5)^3 - (-5)^2 + (-5) + 1 = -125 - 25 - 5 + 1 = -154$, $h(0) = (0)^3 - (0)^2 + 0 + 1 = 1$, $h(a) = a^3 - (a)^2 + a + 1 = a^3 - a^2 + a + 1$, and $h(-a) = (-a)^3 - (-a)^2 + (-a) + 1 = -a^3 - a^2 - a + 1$.

5. $f(x) = 2x + 5$, so $f(a+h) = 2(a+h) + 5 = 2a + 2h + 5$, $f(-a) = 2(-a) + 5 = -2a + 5$, $f\left(a^2\right) = 2\left(a^2\right) + 5 = 2a^2 + 5$, $f(a - 2h) = 2(a - 2h) + 5 = 2a - 4h + 5$, and $f(2a - h) = 2(2a - h) + 5 = 4a - 2h + 5$

6. $g(x) = -x^2 + 2x$, $g(a+h) = -(a+h)^2 + 2(a+h) = -a^2 - 2ah - h^2 + 2a + 2h$, $g(-a) = -(-a)^2 + 2(-a) = -a^2 - 2a = -a(a+2)$, $g(\sqrt{a}) = -(\sqrt{a})^2 + 2(\sqrt{a}) = -a + 2\sqrt{a}$, $a + g(a) = a - a^2 + 2a = -a^2 + 3a = -a(a-3)$, and $\dfrac{1}{g(a)} = \dfrac{1}{-a^2 + 2a} = -\dfrac{1}{a(a-2)}$.

7. $s(t) = \dfrac{2t}{t^2 - 1}$. Therefore, $s(4) = \dfrac{2(4)}{(4)^2 - 1} = \dfrac{8}{15}$, $s(0) = \dfrac{2(0)}{0^2 - 1} = 0$, $s(a) = \dfrac{2(a)}{a^2 - 1} = \dfrac{2a}{a^2 - 1}$; $s(2+a) = \dfrac{2(2+a)}{(2+a)^2 - 1} = \dfrac{2(2+a)}{a^2 + 4a + 4 - 1} = \dfrac{2(2+a)}{a^2 + 4a + 3}$, and $s(t+1) = \dfrac{2(t+1)}{(t+1)^2 - 1} = \dfrac{2(t+1)}{t^2 + 2t + 1 - 1} = \dfrac{2(t+1)}{t(t+2)}$.

8. $g(u) = (3u - 2)^{3/2}$. Therefore, $g(1) = [3(1) - 2]^{3/2} = (1)^{3/2} = 1$, $g(6) = [3(6) - 2]^{3/2} = 16^{3/2} = 4^3 = 64$, $g\left(\frac{11}{3}\right) = \left[3\left(\frac{11}{3}\right) - 2\right]^{3/2} = (9)^{3/2} = 27$, and $g(u+1) = [3(u+1) - 2]^{3/2} = (3u+1)^{3/2}$.

9. $f(t) = \dfrac{2t^2}{\sqrt{t-1}}$. Therefore, $f(2) = \dfrac{2\left(2^2\right)}{\sqrt{2-1}} = 8$, $f(a) = \dfrac{2a^2}{\sqrt{a-1}}$, $f(x+1) = \dfrac{2(x+1)^2}{\sqrt{(x+1)-1}} = \dfrac{2(x+1)^2}{\sqrt{x}}$, and $f(x-1) = \dfrac{2(x-1)^2}{\sqrt{(x-1)-1}} = \dfrac{2(x-1)^2}{\sqrt{x-2}}$.

10. $f(x) = 2 + 2\sqrt{5-x}$. Therefore, $f(-4) = 2 + 2\sqrt{5-(-4)} = 2 + 2\sqrt{9} = 2 + 2(3) = 8$, $f(1) = 2 + 2\sqrt{5-1} = 2 + 2\sqrt{4} = 2 + 4 = 6$, $f\left(\frac{11}{4}\right) = 2 + 2\left(5 - \frac{11}{4}\right)^{1/2} = 2 + 2\left(\frac{9}{4}\right)^{1/2} = 2 + 2\left(\frac{3}{2}\right) = 5$, and $f(x+5) = 2 + 2\sqrt{5-(x+5)} = 2 + 2\sqrt{-x}$.

11. Because $x = -2 \le 0$, we calculate $f(-2) = (-2)^2 + 1 = 4 + 1 = 5$. Because $x = 0 \le 0$, we calculate $f(0) = (0)^2 + 1 = 1$. Because $x = 1 > 0$, we calculate $f(1) = \sqrt{1} = 1$.

12. Because $x = -2 < 2$, $g(-2) = -\frac{1}{2}(-2) + 1 = 1 + 1 = 2$. Because $x = 0 < 2$, $g(0) = -\frac{1}{2}(0) + 1 = 0 + 1 = 1$. Because $x = 2 \ge 2$, $g(2) = \sqrt{2-2} = 0$. Because $x = 4 \ge 2$, $g(4) = \sqrt{4-2} = \sqrt{2}$.

13. Because $x = -1 < 1$, $f(-1) = -\frac{1}{2}(-1)^2 + 3 = \frac{5}{2}$. Because $x = 0 < 1$, $f(0) = -\frac{1}{2}(0)^2 + 3 = 3$. Because $x = 1 \ge 1$, $f(1) = 2\left(1^2\right) + 1 = 3$. Because $x = 2 \ge 1$, $f(2) = 2\left(2^2\right) + 1 = 9$.

14. Because $x = 0 \le 1$, $f(0) = 2 + \sqrt{1-0} = 2 + 1 = 3$. Because $x = 1 \le 1$, $f(1) = 2 + \sqrt{1-1} = 2 + 0 = 2$. Because $x = 2 > 1$, $f(2) = \frac{1}{1-2} = \frac{1}{-1} = -1$.

15. **a.** $f(0) = -2$.

b. **(i)** $f(x) = 3$ when $x \approx 2$. **(ii)** $f(x) = 0$ when $x = 1$.

c. $[0, 6]$

d. $[-2, 6]$

16. **a.** $f(7) = 3$. **b.** $x = 4$ and $x = 6$. **c.** $x = 2$; 0. **d.** $[-1, 9]$; $[-2, 6]$.

17. $g(2) = \sqrt{2^2 - 1} = \sqrt{3}$, so the point $\left(2, \sqrt{3}\right)$ lies on the graph of g.

18. $f(3) = \frac{3+1}{\sqrt{3^2+7}} + 2 = \frac{4}{\sqrt{16}} + 2 = \frac{4}{4} + 2 = 3$, so the point $(3, 3)$ lies on the graph of f.

19. $f(-2) = \frac{|-2-1|}{-2+1} = \frac{|-3|}{-1} = -3$, so the point $(-2, -3)$ does lie on the graph of f.

20. $h(-3) = \frac{|-3+1|}{(-3)^3+1} = \frac{2}{-27+1} = -\frac{2}{26} = -\frac{1}{13}$, so the point $\left(-3, -\frac{1}{13}\right)$ does lie on the graph of h.

21. Because the point $(1, 5)$ lies on the graph of f it satisfies the equation defining f. Thus, $f(1) = 2(1)^2 - 4(1) + c = 5$, or $c = 7$.

22. Because the point $(2, 4)$ lies on the graph of f it satisfies the equation defining f. Thus, $f(2) = 2\sqrt{9-(2)^2} + c = 4$, or $c = 4 - 2\sqrt{5}$.

23. Because $f(x)$ is a real number for any value of x, the domain of f is $(-\infty, \infty)$.

24. Because $f(x)$ is a real number for any value of x, the domain of f is $(-\infty, \infty)$.

25. $f(x)$ is not defined at $x = 0$ and so the domain of f is $(-\infty, 0) \cup (0, \infty)$.

26. $g(x)$ is not defined at $x = 1$ and so the domain of g is $(-\infty, 1) \cup (1, \infty)$.

27. $f(x)$ is a real number for all values of x. Note that $x^2 + 1 \ge 1$ for all x. Therefore, the domain of f is $(-\infty, \infty)$.

28. Because the square root of a number is defined for all real numbers greater than or equal to zero, we have $x - 5 \ge 0$ or $x \ge 5$, and the domain is $[5, \infty)$.

29. Because the square root of a number is defined for all real numbers greater than or equal to zero, we have $5 - x \ge 0$, or $-x \ge -5$ and so $x \le 5$. (Recall that multiplying by -1 reverses the sign of an inequality.) Therefore, the domain of f is $(-\infty, 5]$.

30. Because $2x^2 + 3$ is always greater than zero, the domain of g is $(-\infty, \infty)$.

31. The denominator of f is zero when $x^2 - 1 = 0$, or $x = \pm 1$. Therefore, the domain of f is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

32. The denominator of f is equal to zero when $x^2+x-2=(x+2)(x-1)=0$; that is, when $x=-2$ or $x=1$. Therefore, the domain of f is $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$.

33. f is defined when $x+3\geq 0$, that is, when $x\geq -3$. Therefore, the domain of f is $[-3,\infty)$.

34. g is defined when $x-1\geq 0$; that is when $x\geq 1$. Therefore, the domain of f is $[1,\infty)$.

35. The numerator is defined when $1-x\geq 0$, $-x\geq -1$ or $x\leq 1$. Furthermore, the denominator is zero when $x=\pm 2$. Therefore, the domain is the set of all real numbers in $(-\infty,-2)\cup(-2,1]$.

36. The numerator is defined when $x-1\geq 0$, or $x\geq 1$, and the denominator is zero when $x=-2$ and when $x=3$. So the domain is $[1,3)\cup(3,\infty)$.

37. a. The domain of f is the set of all real numbers.

b. $f(x)=x^2-x-6$, so

$f(-3)=(-3)^2-(-3)-6=9+3-6=6$,

$f(-2)=(-2)^2-(-2)-6=4+2-6=0$,

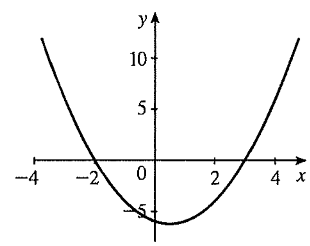
$f(-1)=(-1)^2-(-1)-6=1+1-6=-4$,

$f(0)=(0)^2-(0)-6=-6$,

$f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)-6=\frac{1}{4}-\frac{2}{4}-\frac{24}{4}=-\frac{25}{4}$, $f(1)=(1)^2-1-6=-6$,

$f(2)=(2)^2-2-6=4-2-6=-4$, and $f(3)=(3)^2-3-6=9-3-6=0$.

c.



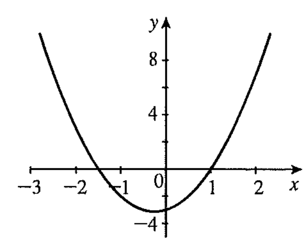
38. $f(x)=2x^2+x-3$.

a. Because $f(x)$ is a real number for all values of x, the domain of f is $(-\infty,\infty)$.

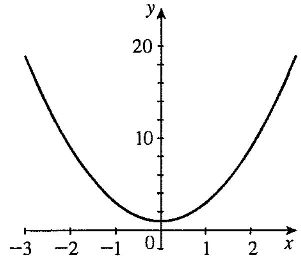
b.

x	-3	-2	-1	$-\frac{1}{2}$	0	1	2	3
y	12	3	-2	-3	-3	0	7	18

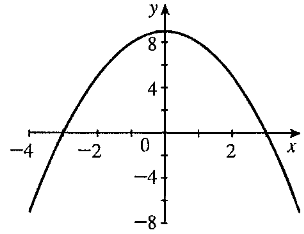
c.



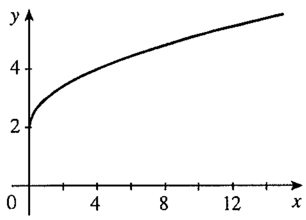
39. $f(x)=2x^2+1$ has domain $(-\infty,\infty)$ and range $[1,\infty)$.



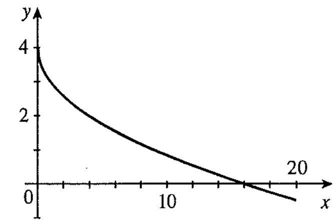
40. $f(x)=9-x^2$ has domain $(-\infty,\infty)$ and range $(-\infty,9]$.



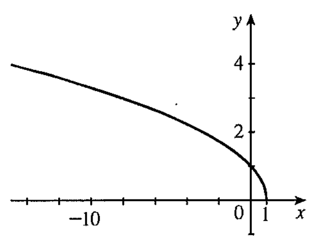
41. $f(x) = 2 + \sqrt{x}$ has domain $[0, \infty)$ and range $[2, \infty)$.



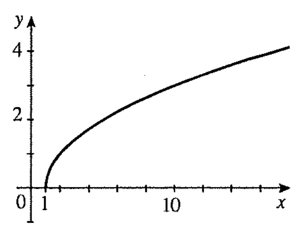
42. $g(x) = 4 - \sqrt{x}$ has domain $[0, \infty)$ and range $(-\infty, 4]$.



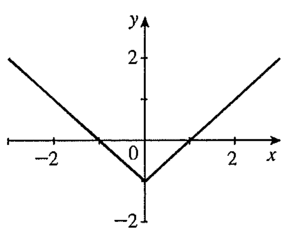
43. $f(x) = \sqrt{1 - x}$ has domain $(-\infty, 1]$ and range $[0, \infty)$



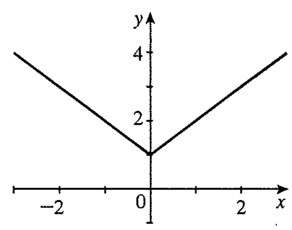
44. $f(x) = \sqrt{x - 1}$ has domain $(1, \infty)$ and range $[0, \infty)$.



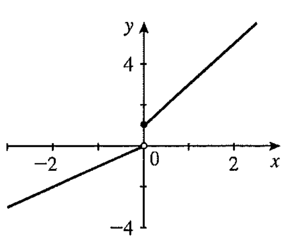
45. $f(x) = |x| - 1$ has domain $(-\infty, \infty)$ and range $[-1, \infty)$.



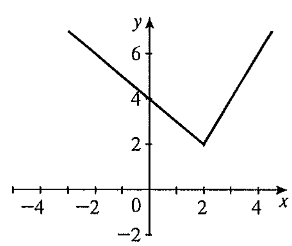
46. $f(x) = |x| + 1$ has domain $(-\infty, \infty)$ and range $[1, \infty)$



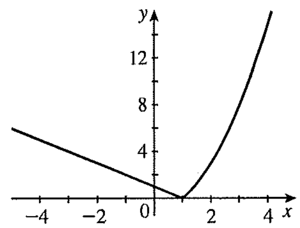
47. $f(x) = \begin{cases} x & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$ has domain $(-\infty, \infty)$ and range $(-\infty, 0) \cup [1, \infty)$.



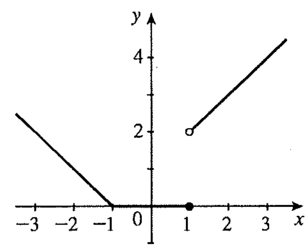
48. For $x < 2$, the graph of f is the half-line $y = 4 - x$. For $x \geq 2$, the graph of f is the half-line $y = 2x - 2$. f has domain $(-\infty, \infty)$ and range $[2, \infty)$.



49. If $x \le 1$, the graph of f is the half-line $y = -x + 1$. For $x > 1$, we calculate a few points: $f(2) = 3$, $f(3) = 8$, and $f(4) = 15$. f has domain $(-\infty, \infty)$ and range $[0, \infty)$.



50. If $x < -1$ the graph of f is the half-line $y = -x - 1$. For $-1 \le x \le 1$, the graph consists of the line segment $y = 0$. For $x > 1$, the graph is the half-line $y = x + 1$. f has domain $(-\infty, \infty)$ and range $[0, \infty)$.



51. Each vertical line cuts the given graph at exactly one point, and so the graph represents y as a function of x.

52. Because the y-axis, which is a vertical line, intersects the graph at two points, the graph does not represent y as a function of x.

53. Because there is a vertical line that intersects the graph at three points, the graph does not represent y as a function of x.

54. Each vertical line intersects the graph of f at exactly one point, and so the graph represents y as a function of x.

55. Each vertical line intersects the graph of f at exactly one point, and so the graph represents y as a function of x.

56. The y-axis intersects the circle at *two* points, and this shows that the circle is not the graph of a function of x.

57. Each vertical line intersects the graph of f at exactly one point, and so the graph represents y as a function of x.

58. A vertical line containing a line segment comprising the graph cuts it at infinitely many points and so the graph does not define y as a function of x.

59. The circumference of a circle with a 5-inch radius is given by $C(5) = 2\pi(5) = 10\pi$, or 10π inches.

60. $V(2.1) = \frac{4}{3}\pi(2.1)^3 \approx 38.79$, $V(2) = \frac{4}{3}\pi(8) \approx 33.51$, and so $V(2.1) - V(2) = 38.79 - 33.51 = 5.28$ is the amount by which the volume of a sphere of radius 2.1 exceeds the volume of a sphere of radius 2.

61. $C(0) = 6$, or 6 billion dollars; $C(50) = 0.75(50) + 6 = 43.5$, or 43.5 billion dollars; and $C(100) = 0.75(100) + 6 = 81$, or 81 billion dollars.

62. The child should receive $D(4) = \frac{2}{25}(500)(4) = 160$, or 160 mg.

63. **a.** From $t = 0$ through $t = 5$, that is, from the beginning of 2001 until the end of 2005.

b. From $t = 5$ through $t = 9$, that is, from the beginning of 2006 until the end of 2010.

c. The average expenditures were the same at approximately $t = 5.2$, that is, in the year 2006. The level of expenditure on each service was approximately \$900.