Chapter 3.6 – Implicit Differentiation and Related Rates

The function $y = f(x)$ is said to be in *explicit* form as y is explicitly expressed in terms of x. The "y" is isolated on the left of the equal sign and is considered to be on the "outside" of the function. (Think **EX**ternal = **EX**plicit). The following is an example of an explicit equation:

$$y = \frac{x^2}{1-x}$$

Consider the example:

$$x^2y + 1 - x^2 + y = 0$$

This equation is in *implicit* form. That is, the "y" is mixed up inside of the equation. (Think **I**nternal = **Im**plicit).

IMPLICIT DIFFERENTIATION

Implicit differentiation is done when equations are given in implicit form, that is, when "y" is not isolated to one side of the equal sign and uses the notation $\frac{dy}{dx}$.

$\frac{dy}{dx}$ is another way to represent the derivative. One interpretation of $\frac{dy}{dx}$ is "the change in y with respect to x". In other words, how y is changing with respect to how x is changing. Previously the derivative y' was written in terms of "x". Here "x" is seen as the "important" variable. With implicit differentiation, x will still be seen as the important variable – the one you find the derivative of.

EXAMPLE 2:

$$x^2 + y^2 = 6$$

Notice that "y" is not isolated on one side of the equal sign. To find the derivative, we will find the derivative of the first term:

Find the derivative:

1. $x^3 + y^3 + y - 4 = 0$

$$3x^2 + 3y^2\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$(3y^2+1)\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{(3y^2+1)}$$

2. $x^2y^3 - 2xy^2 = 5$

$$y^3(2x) + x^2\left(3y^2\frac{dy}{dx}\right) - 2\left[y^2(1) + x\left(2y\frac{dy}{dx}\right)\right] = 0$$

$$2xy^3 + 3x^2y^2\frac{dy}{dx} - 2y^2 - 4xy\frac{dy}{dx} = 0$$

$$(3x^2y^2 - 4xy)\frac{dy}{dx} = 2y^2 - 2xy^3$$

$$\frac{dy}{dx} = \frac{2y^2(1-xy)}{xy(3xy-4)} = \boxed{\frac{2y(1-xy)}{x(3xy-4)}}$$

d. $A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

e. $\frac{dA}{dt} = 2\pi r\left(2 \frac{cm}{min}\right) = (4\pi)(r) \frac{cm}{min} = \boxed{160\pi \frac{cm^2}{min}}$ when $r = 40$ cm

Solve the following related rates:

3. A spherical ball of snow is melting at a rate of $72\pi\ ft^3/\sec$ (its volume is decreasing at that rate). At the moment that its diameter is 6ft, how fast is the radius decreasing?

4. A 40-foot ladder that is leaning against a wall begins to slide. How fast is the bottom of the ladder sliding away from the wall at the time when the top of the ladder is sliding down the wall at a rate of 9 ft/sec and the bottom of the ladder is 24 feet from the wall? [Hint: Use Pythagoras' Theorem to form a relationship between variables and consider movement down the wall to be negative.]

5. Air is being pumped into a spherical balloon at a rate of 5 cubic centimetres per minute. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20cm.

6. A 17-ft ladder (much more realistic) is resting against a wall. Jake is balancing precariously at the very top of the ladder. The bottom of the ladder is 8 feet away from the wall, so you decide to kick the ladder away from the house at a rate of ¼ ft/sec. How fast is Jake falling? Why are you and Jake no longer friends?

7. A screen saver displays the outline of a 3cm by 2cm rectangle and then expands the rectangle in such a way that the 2cm side is expanding at a rate of 4cm/sec and the proportions of the rectangle never change. How fast is the area of the rectangle increasing when its dimensions are 12cm by 8cm?

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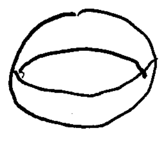
3.6 Notes Problems

Notes Ch 3.6 Implicit Differentiation and Related Rates

(Problems on Notes)

3. $\frac{dV}{dt} = 72\pi \frac{ft^3}{s}$



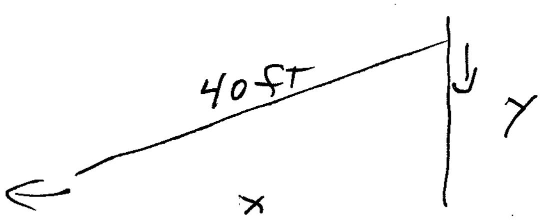
$V = \frac{4\pi r^3}{3}$

$d = 6ft$ so $r = 3ft$ find $\frac{dr}{dt}$

$V = \frac{4\pi r^3}{3}$ so $\frac{dV}{dt} = \frac{(\cancel{3})4\pi r^2}{\cancel{3}}\frac{dr}{dt}$

so $\frac{dr}{dt} = \frac{\left(\frac{dV}{dt}\right)}{4\pi r^2} = \frac{72\cancel{\pi}\frac{ft^3}{s}}{4\cancel{\pi}\underbrace{\cancel{(3ft)^2}}_{9ft^2}} = \boxed{2\frac{ft}{s}}$

4.



$x^2 + y^2 = 1600 ft^2$
$24^2 + y^2 = 1600$
$y^2 = 1600 - 576 = 1024$
$y = 32$ ft

$\frac{dx}{dt} = ?$ $\quad \frac{dy}{dt} = 9\frac{ft}{sec}$; $x = 24 ft$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ so $\frac{dx}{dt} = \frac{-\cancel{2}y\left(\frac{dy}{dt}\right)}{\cancel{2}x} = -\frac{\overset{4}{\cancel{32}}\cancel{ft}\left(9\frac{ft}{sec}\right)}{\underset{3}{\cancel{24}}\cancel{ft}}$

$\boxed{\frac{dx}{dt} = -12\frac{ft}{sec}}$

Note: means sliding to the left

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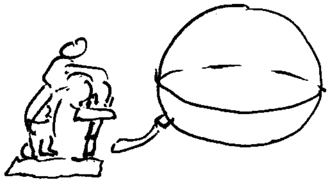
3.6 Notes (cont.)

5.



$V = \frac{4\pi r^3}{3}$

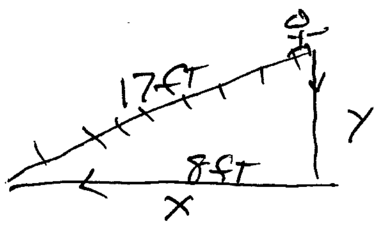
$\frac{dV}{dt} = 5 \frac{cm^3}{min}$; $\frac{dr}{dt} = ?$; when $d = 20cm$, so $r = 10cm$

$$\frac{dV}{dt} = (3)\frac{4\pi}{\cancel{3}}\cancel{3} r^2 \frac{dr}{dt}$$

$$\text{so } \frac{dr}{dt} = \frac{\left(\frac{dV}{dt}\right)}{4\pi r^2} = \frac{\cancel{5}^{1} \frac{cm^3}{min}}{4\pi (10cm)^2} = \frac{1}{80\pi} \frac{cm}{min} \doteq \boxed{.004 \frac{cm}{min}}$$

$\cancel{100}cm^2$ 20

6.



$x^2 + y^2 = 17^2$

$8^2 + y^2 = 17^2$

$y^2 = 17^2 - 8^2$

$y^2 = 289 - 64 = 225$

$y = 15 ft$

when $x = 8ft$; $y = 15ft$

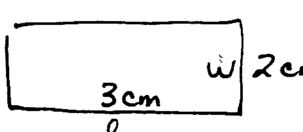
$\frac{dx}{dt} = \frac{1}{4} \frac{ft}{sec}$ $\quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ $\quad \frac{dy}{dt} = ?$

$$\frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = -\frac{x}{y}\frac{dx}{dt} = -\frac{\cancel{8}^{2} ft\left(\frac{1}{\cancel{4}} \frac{ft}{sec}\right)}{15ft}$$

$$\frac{dy}{dt} = -\frac{2}{15}\frac{ft}{sec} = \boxed{-.1\overline{3} \frac{ft}{sec}}$$

Note: neg. sign means falling

7.



$\frac{dw}{dt} = 4 \frac{cm}{sec}$

$\frac{dA}{dt} = ?$ when $l = 12cm$, $w = 8cm$

$A = lw$ $\quad \frac{dw/dt}{dl/dt} = \frac{2}{3}$ so $\frac{4}{\frac{dl}{dt}} = \frac{2}{3}$ so $\frac{dl}{dt} = 6$

$$\frac{dA}{dt} = w\frac{dl}{dt} + l\frac{dw}{dt} = (8cm)\left(6\frac{cm}{sec}\right) + (12cm)\left(4\frac{cm}{sec}\right)$$

$$\frac{dA}{dt} = 48\ cm^2/sec + 48\ cm^2/sec = \boxed{96 \frac{cm^2}{sec}}$$

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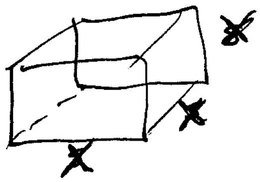
Hmwk 3.6 #50

50. Volume of a Cube



$V = x^3$ $\qquad \frac{dV}{dt} = ?$

when $x = 5\,in$; $V = 125\,in^3$; $\frac{dx}{dt} = 0.1\ \frac{inches}{sec}$

find $\frac{dV}{dt}$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} = 3\,\overset{25\,in^2}{(5_{in})^2}\left(0.1\,\frac{in}{sec}\right) = \boxed{7.5\ \frac{in^3}{sec}}$$