Key

Chapter 6.1 – Antiderivatives and the Rules of Integration

Formal Definition:

A function F is an antiderivative of *f* on an interval I if $F'(x) = f(x) \ \forall\ x$

Informal Definition:

The antiderivative is the function that results from "undoing" the derivative. In other words, it would represent the original function, that when differentiated, gives you the answer. For example consider:

$$F(x) = x^3 \ \textit{is an antiderivative of} \ f(x) = 3x^2$$

That is, we try to figure out the function whose derivative produces f(x). We are undoing the derivative, moving in the opposite direction – from the derivative to the original function.

Try finding an antiderivative, F(x), of the examples below:

1. $f(x) = 2x + 6$ $\longrightarrow$ $x^2 + 6x$
2. $f(x) = 3x^2 + 2x + 7$ $x^3 + x^2 + 7x$
3. $f(x) = 4x^3 - 9$ $x^4 - 9x$
4. $f(x) = 5x^4 + 2x$ $x^5 + x^2$
5. $f(x) = 6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$ $x^6 + x^5 + x^4 + x^3 + x^2 + x$
6. $f(x) = 4x$ $2x^2$
7. $f(x) = 9x^2$ $3x^3$
8. $f(x) = 12x^3 - 6x^2 + 8x - 5$ $3x^4 - 2x^3 + 4x^2 - 5x$

Note that we talk about "an" antiderivative vs "the" antiderivative. The difference will now be explained.

Basic Integration Rules:

Differentiation (the derivative) and antidifferentiation (integration/the integral) are reverse operations. In the last exercise, we tried to determine a possible original function when we were given the derivative. For example:

An antiderivative of $5x^4$ is x^5.

What we should notice is that the answer could also have been x^5+7 or x^5-127 or $x^5+986{,}124$. They are all referred to as "an" antiderivative. In other words, there are several, equally correct solutions. This is referred to as finding the **indefinite integral** (think: no definite solution/several solutions).

Rule 1: The Indefinite Integral of a Constant

$$\int k\ dx = kx + C, \qquad k = a\ \text{constant}$$

To prove the above rule you need only verify that the derivative of kx + C = k. The "dx" indicates the important variable – the one we find the derivative or the integral with respect to.

<u>EXAMPLE 1:</u>

Find the indefinite integral:

$$\int 3\ dx = 3x + C$$

$$\int e^2\ dx = e^2 + C$$

Key

Rule 2: The Power Rule

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$$

EXAMPLE 2:

Find the integral of:

$$\int x^4 \, dx = \frac{1}{5} x^5 + C$$

$$\int x^{\frac{5}{2}} \, dx = \frac{1}{\left(\frac{5}{2}+1\right)} x^{\frac{7}{2}} + C = \boxed{\frac{2}{7} x^{7/2} + C}$$

$$\int \frac{1}{x^{\frac{3}{2}}} \, dx = \int x^{-3/2} dx = \frac{1}{\left(-\frac{3}{2}+1\right)} x^{(-1/2)} + C = \boxed{-2x^{-\frac{1}{2}} + C}$$

OR $\boxed{-\frac{2}{\sqrt{x}} + C}$

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int c \cdot f(x) \, dx = c \int f(x) dx, \quad c = a \; constant$$

EXAMPLE 3:

Find the integral of:

$$\int 6r^2 \, dr = 6\left(\frac{1}{3}\right) r^3 + C = \boxed{2r^3 + C}$$

$$\int -4x^{-3} \, dx = -4\left(\frac{1}{-2}\right) x^{-2} + C = \boxed{2x^{-2} + C} \text{ OR } \boxed{\frac{2}{x^2} + C}$$

Rule 4: The Sum/Difference Rule

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$$

<u>EXAMPLE 4:</u>

$$\int \left(3x^5 + 4x^{\frac{3}{2}} - 2x^{\frac{-1}{2}}\right)dx = \frac{1}{2}x^6 + 4\left(\frac{1}{\frac{5}{2}}\right)x^{5/2} - 2\left(\frac{1}{\frac{1}{2}}\right)x^{1/2} + C$$

$$= \boxed{\frac{1}{2}x^6 + \frac{8}{5}x^{5/2} - 4x^{1/2} + C}$$

Rule 5: The Indefinite Integral of the Exponential Function

$$\int e^x \, dx = e^x + C$$

<u>EXAMPLE 5:</u>

$$\int \left(2e^x - 3x^2\right)dx = 2e^x - x^3 + C$$

Rule 6: The Indefinite Integral of the Function $f(x) = x^{-1}$

$$\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C, \quad x \neq 0$$

<u>EXAMPLE 6:</u>

$$\int 2x^{-2}dx \Downarrow$$

$$\int \left(4x + \frac{3}{x} + \frac{2}{x^2}\right)dx = \boxed{2x^2 + 3\ln|x| - 2x^{-1} + C}$$

OR $\boxed{2x^2 + 3\ln|x| - \frac{2}{x} + C}$

ANTIDERIVATIVES

$\boxed{F(x)}$

Find F(x), the antiderivative, of the following functions:

1. $f(x) = x - 3$ $\quad \boxed{F(x) = \frac{1}{2}x^2 - 3x}$

2. $f(x) = \frac{1}{2}x^2 - 2x + 6$ $\quad \boxed{\frac{1}{6}x^3 - x^2 + 6x}$

3. $f(x) = 8x^9 - 3x^6 + 12x^3$ $\quad \boxed{\frac{4}{5}x^{10} - \frac{3}{7}x^7 + 3x^4}$

4. $f(x) = (x+1)(2x-1) = 2x^2 + x - 1$ $\quad \boxed{\frac{2}{3}x^3 + \frac{1}{2}x^2 - x}$

5. $f(x) = x(2-x)^2 = x(4 - 4x + x^2) = 4x - 4x^2 + x^3$

$\boxed{2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4}$

6. $f(x) = 5x^{\frac{1}{4}} - 7x^{\frac{3}{4}}$

$5\left(\frac{4}{5}\right)x^{5/4} - 7\left(\frac{4}{7}\right)x^{7/4} = 4x^{5/4} - 4x^{7/4}$

$\boxed{4x^{5/4}(1 - x^{1/2})}$

7. $f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4} = x^{3/4} + x^{4/3}$

$\frac{21}{12}$ $\quad \frac{28}{12}$

$\boxed{\frac{4}{7}x^{7/4} + \frac{3}{7}x^{7/3}} = \frac{1}{7}x^{7/4}\left(4 + 3x^{7/12}\right)$

8. $f(x) = \frac{5 - 4x^3 + 2x^6}{x^6} = 5x^{-6} - 4x^{-3} + 2$

OR →

$-\frac{\cancel{5}}{\cancel{5}}x^{-5} + 2x^{-2} + 2x$

#8. $\boxed{-\frac{1}{x^5} + \frac{2}{x^2} + 2x}$

9. $f(x) = \frac{x^4 + 3\sqrt{x}}{x^2} = \frac{x^4 + 3x^{1/2}}{x^2}$

$f(x) = x^2 + 3x^{-3/2}$

$\frac{1}{3}x^3 + 3(-2)x^{-1/2} = \boxed{\frac{1}{3}x^3 - \frac{6}{\sqrt{x}}}$

Find f(x) if:

$\int\int (6x+12x^2)$

10. $f''(x)=6x+12x^2$ $f'(x)=3x^2+4x^3$ $\boxed{f(x)=x^3+x^4}$

11. $f''(x)=\frac{2}{3}x^{\frac{2}{3}}$ $f'(x)=\frac{2}{3}\left(\frac{3}{5}\right)x^{5/3}=\frac{2}{5}x^{5/3}$

$f(x)=\frac{\cancel{2}^{1}}{5}\left(\frac{3}{\cancel{8}_{4}}\right)x^{8/3}$

11. $f(x)=\frac{3}{20}x^{8/3}$

Checks

12. $f''(x)=2+x^3+x^6$

$f'(x)=2x+\frac{1}{4}x^4+\frac{1}{7}x^7$

$f(x)=x^2+\frac{1}{20}x^5+\frac{1}{56}x^8$

13. $f'''(x)=60x^2$

$f''(x)=20x^3$ $\boxed{13.\ f(x)=x^5}$

$f'(x)=5x^4$

Use the following information to determine the EXACT formula of the function, f(x):

14. $f'(x)=1-6x, \quad f(0)=8$

$\int(1-6x)dx$

$f(x)=\cancel{f(x)}=x-3x^2+C$

$f(0)=0-3(0)^2+C=8$

$C=8$

so $\boxed{f(x)=x-3x^2+8}$

15. $f'(x)=8x^3+12x+3, \quad f(1)=6$

$f(x)=2x^4+6x^2+3x+C$

$f(1)=2+\cancel{6}+3+C=\cancel{6}$

$C=-5$

so $\boxed{f(x)=2x^4+6x^2+3x-5}$

Chapter 6.2 – Integration by Substitution

So far we have performed integrals where the derivatives were in standard, straightforward forms. Now we will consider doing the integral of functions that are more complicated – namely when the form involves a chain rule. Consider:

$$\int 3(3x+2)^5\,dx$$

We could find the integral by expanding and then integrating each resulting term. This method, though, is too tedious. Instead, we will integrate by **substitution**. This involves making a change in the variables. Namely,

$$u = 3x+2$$

Then finding the derivative we get,

$$du = 3\,dx \qquad \left(\frac{du}{dx} = 3\right)$$

Substituting into the original equation, we obtain

$$\int 3(3x+2)^5\,dx = \int (3x+2)^5 \cdot 3dx = \int u^5\;du$$

The integral is reduced to a simple power rule which is easy to find:

$$\int u^5 du = \frac{1}{6}u^6 + C$$

Now, replacing u with its equivalent results in:

$$\frac{1}{6}(3x+2)^6 + C$$

Always remember to write the integral in terms of the original variable.

Steps for Integration by Substitution (Affectionately known as "u-substitution")

a. Let u = g(x), where g(x) is part of the integrand, usually the "inside function" of the composition of f(g(x)).
b. Find du = g'(x) dx.
c. Use the substitution u = g(x) and du = g'(x) dx to convert the ENTIRE integral into one involving only u.
d. Evaluate the resulting integral.
e. Replace u by g(x) to obtain the final solution as a function of x.

Be careful when choosing the function to substitute for u. Sometimes the function is not always obvious and the function might need some slight manipulation before or after the substitution is done.

Find the indefinite integral of the following:

1. $\int 4x(4x^2+1)^7\,dx$

let $u = 4x^2+1$

$du = 8x\,dx$ so $\frac{1}{2}du = 4x\,dx$ gives $\frac{1}{2}\int u^7 du = \frac{1}{16}u^8 + C$

$\boxed{= \frac{1}{16}(4x^2+1)^8 + C}$

2. $\int (3x^2-2x+1)(x^3-x^2+x)^4\,dx$

let $u = (x^3 - x^2 + x)$

Then $du = (3x^2 - 2x + 1)dx$ so $\int u^4 du = \frac{1}{5}u^5 + C$

$= \boxed{\frac{1}{5}(x^3 - x^2 + x)^5 + C}$

3. $\int \frac{3x^2+2}{(x^3+2x)^2}\,dx$ let $u = (x^3 + 2x)$

$du = (3x^2+2)dx$

so $\int u^{-2} du = -\frac{1}{1}u^{-1} + C$

$\boxed{= -(x^3+2x)^{-1} + C}$

4. $\int 3t^2(t^3+2)^{\frac{3}{2}}\,dt$ $= \int u^{3/2} du$ OR $= \boxed{-\frac{1}{(x^3+2x)} + C}$

let $u = (t^3+2)$

$du = 3t^2 dt$

$= \frac{2}{5}u^{5/2} + C$

$\boxed{= \frac{2}{5}(t^3+2)^{5/2} + C}$

5. $\int \frac{x^4}{1-x^5}\,dx$

let $u = 1 - x^5$

$du = -5x^4 dx$

$-\frac{1}{5}du = x^4 dx$

$-\frac{1}{5}\int u^{-1} du = -\frac{1}{5}\ln|u| + C$

$\boxed{= -\frac{1}{5}\ln(1-x^5) + C}$

6. $\int \frac{2}{x-2}\,dx$ $= 2\int u^{-1}du = 2\ln|u| + C$

$= \boxed{2\ln(x-2) + C}$

let $u = x-2$

$du = dx$

7. $\int e^{2t+3}\,dt$ let $u = 2t+3$ $\frac{1}{2}\int e^{u}du = \frac{1}{2}e^{u} + C = \boxed{\frac{1}{2}e^{(2t+3)} + C}$

$du = 2dt$

$\frac{1}{2}du = dt$

8. $\int x^2 e^{x^3-1}\,dx$ $= \frac{1}{3}\int e^{u}du = \frac{1}{3}e^{u} + C = \boxed{\frac{1}{3}e^{(x^3-1)} + C}$

let $u = x^3-1$

$du = 3x^2dx$

$\frac{1}{3}du = x^2dx$

9. $\int 3(x^2-1)x\,dx$ $= \int (3x^3 - 3x)\,dx = \boxed{\frac{3}{4}x^4 - \frac{3}{2}x^2 + C}$

10. $\int e^{2x}(e^{2x}+4)^3\,dx$ $= \int u^3du = \frac{1}{4}u^4 + C = \boxed{\frac{1}{4}(e^{2x}+4)^4 + C}$

let $u = e^{2x}+4$

$du = e^{2x}dx$

11. $\int \frac{x}{3x^2-1}\,dx$ $= \frac{1}{6}\int u^{-1}du = \frac{1}{6}\ln|u| + C$

$= \boxed{\frac{1}{6}\ln(3x^2-1) + C}$

let $u = 3x^2-1$

$du = 6x\,dx$

$\frac{1}{6}du = x\,dx$

12. $\int e^{3x}\,dx$

13. $\int e^{-x}dx$ let $u=-x$, $du=-dx$, $-du=dx$

$\int e^{-x}dx = -\int e^{u}du = -e^{u}+C = \boxed{-e^{-x}+C}$

14. $\int e^{2-x}dx$ let $u=2-x$, $-du=dx$

$-\int e^{u}du = -e^{u}+C = \boxed{-e^{(2-x)}+C}$

15. $\int \frac{dx}{2x+3}$ let $u=2x+3$, $du=2dx$, $\frac{1}{2}du=dx$

$\frac{1}{2}\int u^{-1}du = \frac{1}{2}\ln|u|+C = \boxed{\frac{1}{2}\ln|2x+3|+C}$

16. $\int (4-x)^{-1}dx$ let $u=4-x$, $-du=dx$

$-\int u^{-1}du = -\ln|u|+C = \boxed{-\ln|4-x|+C}$

17. $\int \frac{x+2}{x^2}dx$ let $\cancel{u=x^2}$, $\cancel{du=2x}$ doesn't work $= \int (x^{-1}+2x^{-2})dx$

$= \boxed{\ln|x| - 2x^{-1}+C}$

OR $= \boxed{\ln|x| - \frac{2}{x}+C}$

18. *If* $y=e^{3x}-2e^{-3x}$, *show that* $\frac{d^2y}{dx^2}=9y$.

$\frac{dy}{dx} = 3e^{3x}+6e^{-3x}$ so $\frac{d^2y}{dx^2} = 9e^{3x}-18e^{-3x} = 9(e^{3x}-2e^{-3x}) = \boxed{9y}$

19. For a curve, $y=f(x)$, $\frac{d^2y}{dx^2}=6x-2$. Given that $y=11$ *and* $\frac{dy}{dx}=10$ *when* $x=2$, find the equation of the curve.

$\frac{dy}{dx} = \int(6x-2)dx = 3x^2-2x+C$

$3(2)^2-2(2)+C=10$

$12-4+C=10$

$8+C=10$

$C=2$

$f(x) = \int(3x^2-2x+2)dx$

$x^3-x^2+2x+C=11$

$x^3-x^2+2x-11=0$

20. *Given* $\frac{dy}{dx}=1-5x$ *and that* $y=-5$, *when* $x=2$, find the value of y when x = 1.

$y=f(x) = \int(1-5x)dx = x-\frac{5}{2}x^2+C$

$f(2) = 2-\frac{5}{2}(2)^2+C=-5$

$2-10+C=-5$

$C=3$

$f(x) = x-\frac{5}{2}x^2+3$ $\quad f(1)=1-\frac{5}{2}(1)^2+3 = \boxed{\frac{3}{2}}$