Chapter 2.1 – Functions (Review)

Definition:

A function is a rule or "formula" that assigns one element of a set to one and only one element of another set. The most common function notation we will use is:

$$y = f(x)$$

That is, y is a function of x. Here a single x-value (input) is assigned to a single y-value (output). Note that for a function, the reverse does not have to be true.

Domain:

The domain of a function refers to the set of possible input values (x-values). To find the domain of a function, you should keep in mind that:

a) The denominator of a function can NEVER be equal to zero.

b) The even root of a negative number is not real.

Find the domain of the following:

1. $f(x) = \dfrac{1}{x^2 - 9}$
2. $f(x) = x^2 - 7x + 6$
3. $f(x) = \sqrt{x+2}$

Range:

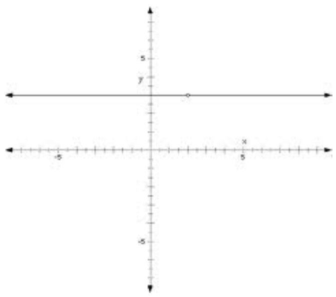
The range of a function refers to the set of possible output values (y-values).

The Vertical Line Test:

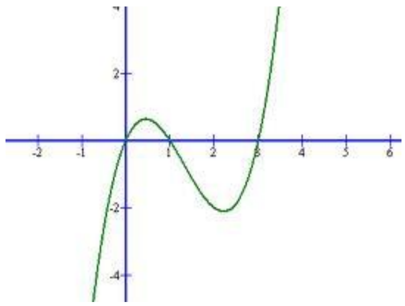
A graph is that of a function iff every vertical line intersects the graph no more than once.

Do the following graphs represent functions?

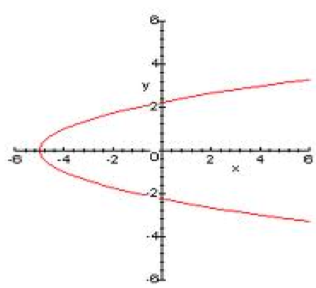
4.



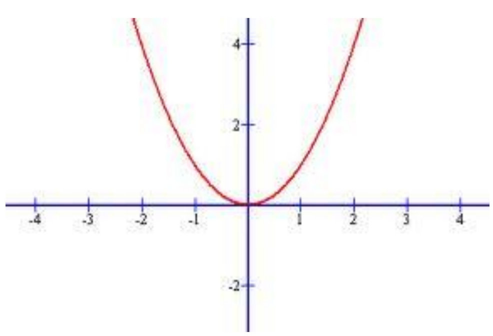
5.



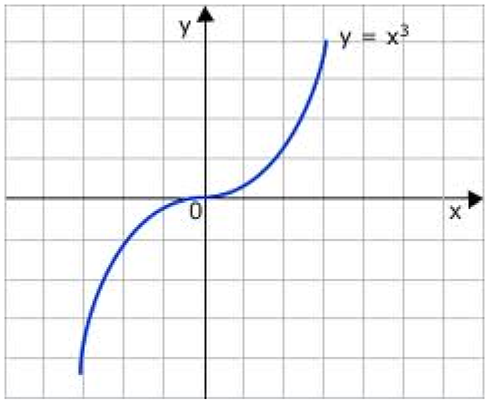
6.



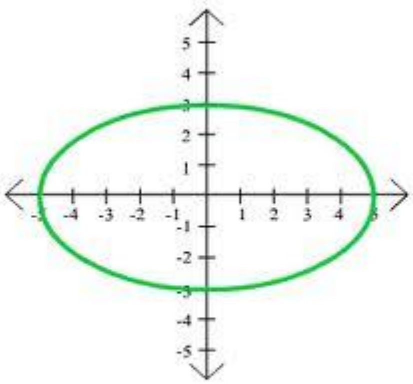
7.



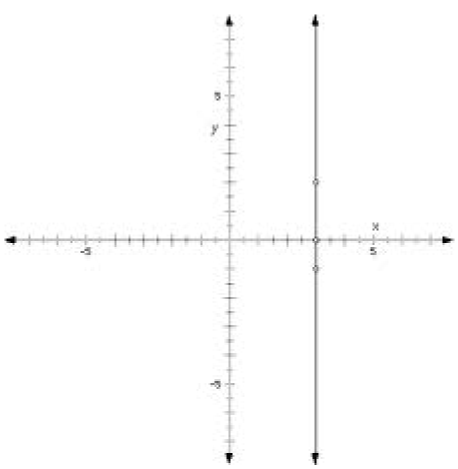
8.



9.



10.



11. The equation of a circle.

Chapter 2.2 – The Algebra of Functions

The Sum of Functions

$$(f+g)(x)=f(x)+g(x)$$

The Difference of Functions

$$(f-g)(x)=f(x)-g(x)$$

The Product of Functions

$$(f\cdot g)(x)=f(x)\cdot g(x)$$

The Quotient of Functions

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)},\ \ g(x)\neq 0$$

The Composition of Functions

$(g\circ f)(x)=g\left[f(x)\right]$ Read "g of f"

Find the following for $f(x)=x-1,\ \ g(x)=x^2+1$

12. $(f+g)(x)$

13. $(f-g)(x)$

14. $(f \cdot g)(x)$

15. $(g \circ f)(x)$

16. $(f \circ g)(x)$

For $f(x) = x + 2,\ g(x) = x^2 - x - 6$

17. $\left(\dfrac{f}{g}\right)(x)$

18. $\left(\dfrac{g}{f}\right)(x)$

Chapter 2.3 – Functions and Mathematical Models

LINEAR EQUATIONS:

Linear equations (equations representing straight lines) have the form:

$$y = mx + b$$

m = slope **(change)** and

b = y-coordinate of the y-intercept **(constant)**

We will look at several situations that can be modeled using a linear equation.

COST, REVENUE, AND PROFIT

The Cost Function, C(x), often has two parts – the *fixed cost* (a cost that occurs no matter what; e.g. the cost of a cellphone plan with a limited number of minutes) and the *variable cost* (a cost that occurs for every unit produced/sold; e.g. the cost associated with going over the limit by x minutes). Note that the variable cost depends on "x".

The Revenue Function, R(x), is simply the price of a unit multiplied by the number of units sold. E.g. the revenue made from selling 6 bags at a price of $10 per bag is 6 x 10 = $60.

The Profit Function, P(x), is the difference between the Revenue Function and the Cost Function. That is P(x) = R(x) – C(x). If your revenue exceeds your costs, you will make a profit, P(x) = +ve (positive). If your revenue is equal to your costs, you "break even"/have no profit, P(x) = 0. If your revenue is less than your costs, you will have a loss, P(x) = -ve (negative).

Example 1:

A company has a fixed cost of $30,000 and a production cost of $6 for each unit it manufactures. A unit sells for $10.

a. What is the cost function?
b. What is the revenue function?
c. What is the profit function?
d. Find the profit (loss) corresponding to production levels of 6000, 8000 and 12000 units.

First, **let x = the number of units sold**. Remember that variables always represent numbers so it would be *incorrect* to let x equal just "units".

a. $C(x)$ = Variable cost + Fixed cost
   $C(x) = 6x + 30{,}000$

b. $R(x) = 10x$

c. $P(x) = R(x) – C(x)$
   $P(x) = 10x – (6x +30{,}000)$
   $P(x) = 10x - 6x - 30{,}000$
   $P(x) = 4x – 30{,}000$

d. (i) $x = 6{,}000$
   $P(x) = P(6{,}000) = 4(6{,}000) – 30{,}000 = -\$6{,}000$ (a loss)

   (ii) $x = 8{,}000$
   $P(x) = P(8{,}000) = 4(8{,}000) – 30{,}000 = \$2{,}000$ (a profit)

   (iii) $x = 12{,}000$
   $P(x) = P(12{,}000) = 4(12{,}000) – 30{,}000 = \$18{,}000$ (a profit)

1. A manufacturer has a production cost of $12 per unit produced and a fixed cost of $30,000. The product sells for $15/unit.
   a. What is the cost function?
   b. What is the revenue function?
   c. What is the profit function?
   d. What profit is expected if 8,000 units are sold?
   e. What profit is expected if 12,000 units are sold?
   f. How many units need to be sold to break even?

Chapter 2.4 – Limits

THE LIMIT OF A FUNCTION: (formal definition)

The function f has a **limit,** *L,* as x approaches a, written

$$\lim_{x \to a} f(x) = L$$

if the value of f(x) can be made as close to the number *L* as we please by taking x sufficiently close to (but not equal to a).

Informal definition:

The limit of a function is the output value (y or f(x)) that the function seems to be approaching as you get close to a particular input value (x). The limit is only about "appearances". It does not matter what the y-value actually is once you get to the x -value – only what seems to be happening.

For example, look at the following function:

$$f(x) = 3x + 2$$

and let us find the limit as x approaches 3, namely

$$\lim_{x \to 3} f(x)$$

Again, the question is not to find f(3). We DO NOT CARE about what the outcome is at x = 3, only about what appears to be the outcome. That means, as we approach 3 from the left (numbers smaller than but very close to 3) and as we approach 3 from the right (numbers larger than but very close to 3), what can be observed?

From the left, e.g. x = 2.999999999, f(2.999999999) = 10.999999997 – which is "close to" 11.

From the right, e.g. x = 3.000000001, f(3.000000001) = 11.000000003 – also "close to" 11.

So, it **seems** that as we approach x = 3, f(x) approaches 11 (from both the left and the right). This is precisely what the limit is! If the output value seems to be the same from the left and the right of an input value, then the limit at the input is that common number. **AGAIN**, the fact that f(3) = 11 has absolutely NOTHING to do with the limit.

Example 2: Consider the piecewise function:

$$h(x) = \begin{cases} x & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

Find $\lim_{x \to 1} h(x)$.

From the left: e.g. x = 0.999999999, h(0.999999999) = 0.999999999 which is close to 1.

From the right: e.g. x = 1.000000001, h(1.000000001) = 2 – 1.000000001 = 0.999999999 which is close to 1.

Therefore: $\lim_{x \to 1} h(x) = 1$

Notice that when x = 1, the value of the function h(x) is actually 0. This value is not the same as the limit of 1, however, we know that the limit at x = 1 has nothing to do with what actually happens at x = 1.

Q: What happens if the left limit is not the same as the right limit?

A: The limit, L, Does Not Exist (DNE).

Q: What happens if the left limit and right limit both approach either positive or negative infinity?

A: The limit, L, again Does Not Exist (DNE).

Q: But even if they both approach, say, positive infinity?

A: Yes the limit still does not exist (DNE). Infinity is not a number you can quantify. If you cannot state the exact value of the limit, then the limit does not exist (DNE).

<u>LIMITS OF FUNCTIONS AT INFINITY:</u>

$$\textit{For all } n > 0,$$

$$\lim_{x\to\infty}\frac{1}{x^n}=0 \quad \textit{and} \quad \lim_{x\to-\infty}\frac{1}{x^n}=0$$

E.g 3. Find $\lim_{x\to\infty}\frac{1}{x^2}$

Since infinity is an extremely large UNKNOWN number, it is impossible to determine numbers to its left and right. Instead, we will just consider infinity itself.

$$\lim_{x\to\infty}\frac{1}{x^2} \;\; \textit{becomes} \;\; \frac{1}{\infty^2}=\frac{1}{\infty}$$

The question now is, why is $\frac{1}{\infty}=0$?

Think about it in terms of pizzas and people. The numerator 1 is the number of pizzas we have to share and ∞ is the number of people to share the pizza with. Just to simplify, let us assume that ∞ is 7 billion, the number of people on earth (7 billion, by the way, is NOT a large number). Now imagine having to equally share 1 pizza with the entire world's population. It is clear that your share will be so close to nothing that we can safely say that you will not be getting any pizza. In other words, your share is 0.

Example 4: Find

$$\lim_{x\to\infty}\frac{x^2+3x+5}{2x^3-6x+7}$$

(Shortcut method)

Step 1: Rewrite the limit including only the terms with the highest powers in both the numerator and denominator.

$$\lim_{x\to\infty}\frac{x^2}{2x^3}$$

Step 2: Simplify the limit, if possible

$$\lim_{x\to\infty}\frac{1}{2x}$$

Steps 3: Substitute x.

$$\frac{1}{2(\infty)}=\frac{1}{\infty}=0$$

So, $\lim_{x\to\infty}\dfrac{x^2+3x+5}{2x^3-6x+7}=0$

Example 5. Find

$$\lim_{x\to-\infty}\frac{3x^5-9x+2}{2x^2+4x}$$

$$\lim_{x\to-\infty}\frac{3x^5-9x+2}{2x^2+4x}=\lim_{x\to-\infty}\frac{3x^5}{2x^2}=\lim_{x\to-\infty}\frac{3x^3}{2}=\lim_{x\to-\infty}\frac{3(-\infty)^3}{2}=-\infty=DNE$$

Find the corresponding limits:

2. $f(x)=x^3,\ \lim_{x\to 3} f(x)$

3. $k(x)=\begin{cases}2+x & if\ \ x\neq 1\\ 1 & if\ \ x=1\end{cases}$

$\lim_{x\to 1} k(x)$

4. $g(x)=\begin{cases} 1 & if\ x<0 \\ -1 & if\ x\geq 0 \end{cases}$

$\lim_{x\to 0} g(x)$

5. $f(x)=\dfrac{1}{x^3},\ \lim_{x\to\infty} f(x)$

6. $\lim_{x\to\infty}\dfrac{x^2-x+3}{2x^3+1}$

7. $\lim_{x\to\infty}\dfrac{3x^2+8x-4}{2x^2+4x-5}$

8. $\lim_{x\to-\infty}\dfrac{2x^3+1-3x^2}{4+x^2+2x}$

9. $\lim_{x\to-1}\dfrac{x^2-x-2}{2x^2-x-3}$

SPECIAL RULES: There are five special cases where direct substitution can be used to calculate limits. We have already encountered some of them.

a. Limits at infinity.
b. For functions that are polynomial.
c. For functions that are rational.
d. For functions that are trigonometric.
e. For functions that are radical.

Find the following limits:

1. $\lim_{x\to 2} 3x-5$

2. $\lim_{x\to 3}\frac{x^2-9}{x-3}$

3. $\lim_{x\to 6}\frac{x^2-4x-12}{x-6}$

4. $\lim_{x\to 3}\sqrt{x^2+16}$

5. $\lim_{x\to 6}\frac{4x+12}{x-6}$

6. $\lim_{x\to 2}\sqrt{5x^3-15}$

7. $\lim_{x \to 1} \frac{2-x}{(x-1)^2}$

8. $\lim_{x \to \infty} \frac{x+x^3+x^5}{1+x^4-x^2}$

9. $\lim_{x \to \infty} \; x^4+x^5$

10. $\lim_{u \to \infty} \frac{4u^2+5}{(u^2-2)(2u^2-1)}$

11. $\lim_{x \to -\infty} \frac{3x+5}{x-4}$

12. $\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$ *

13. $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t}$ *

14. $\lim_{p \to 0} \frac{\sqrt{p+25}-5}{p}$ *