Properties Relating the Exponential and Logarithmic Functions

We made use of the relationship that exists between the exponential function $f(x) = e^x$ and the logarithmic function $g(x) = \ln x$ when we sketched the graph of g in Example 6. This relationship is further described by the following properties, which are an immediate consequence of the definition of the logarithm of a number.

Properties Relating e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{for } x > 0) \qquad (2)$$

$$\ln e^x = x \qquad (\text{for any real number } x) \qquad (3)$$

(Try to verify these properties.)

From Properties 2 and 3, we conclude that the composite function satisfies

$$\begin{aligned}(f \circ g)(x) &= f[g(x)]\\ &= e^{\ln x} = x \qquad (\text{for all } x > 0)\\ (g \circ f)(x) &= g[f(x)]\\ &= \ln e^x = x \qquad (\text{for all } x > 0)\end{aligned}$$

Any two functions f and g that satisfy this relationship are said to be **inverses** of each other. Note that the function f undoes what the function g does, and vice versa, so the composition of the two functions in any order results in the identity function $F(x) = x$.*

The relationships expressed in Equations (2) and (3) are useful in solving equations that involve exponentials and logarithms.

Exploring with TECHNOLOGY

You can demonstrate the validity of Properties 2 and 3, which state that the exponential function $f(x) = e^x$ and the logarithmic function $g(x) = \ln x$ are inverses of each other, as follows:

1. Sketch the graph of $(f \circ g)(x) = e^{\ln x}$, using the viewing window $[0, 10] \times [0, 10]$. Interpret the result.
2. Sketch the graph of $(g \circ f)(x) = \ln e^x$, using the standard viewing window. Interpret the result.

EXAMPLE 7 Solve the equation $2e^{x+2} = 5$.

Solution We first divide both sides of the equation by 2 to obtain

$$e^{x+2} = \frac{5}{2} = 2.5$$

Next, taking the natural logarithm of each side of the equation and using Equation

$$\begin{aligned}\ln e^{x+2} &= \ln 2.5\\ x + 2 &= \ln 2.5\\ x &= -2 + \ln 2.5\\ &\approx -1.08\end{aligned}$$

EXAMPLE 8 Solve the equation $5 \ln x + 3 = 0$.

Solution Adding -3 to both sides of the equation leads to

$$\begin{aligned}5 \ln x &= -3\\ \ln x &= -\frac{3}{5} = -0.6\end{aligned}$$

and so

$$e^{\ln x} = e^{-0.6}$$

Using Equation (2), we conclude that

$$\begin{aligned}x &= e^{-0.6}\\ &\approx 0.55\end{aligned}$$

5.2 Self-Check Exercises

1. Sketch the graph of $y = 3^x$ and $y = \log_3 x$ on the same set of axes.
2. Solve the equation $3e^{x+1} - 2 = 4$.

Solutions to Self-Check Exercises 5.2 can be found on page 354.

5.2 Concept Questions

1. **a.** Define $y = \log_b x$.
   b. Define the logarithmic function f with base b. What restrictions, if any, are placed on b?
2. For the logarithmic function $y = \log_b x$ $(b > 0, b \neq 1)$, state (a) its domain and range, (b) its x-intercept, (c) where it is continuous, and (d) where it is increasing and where it is decreasing for the case $b > 1$ and the case $b < 1$.
3. **a.** If $x > 0$, what is $e^{\ln x}$?
   b. If x is any real number, what is $\ln e^x$?
4. Let $f(x) = \ln x^2$ and $g(x) = 2 \ln x$. Are f and g identical?
   Hint: Look at their domains.

5.2 Exercises

nine each group
Problems

In Exercises 1–10, express each equation in logarithmic form.

1. $2^6 = 64$
2. $3^5 = 243$ $\log_5\left(\frac{1}{125}\right) = -3$
3. $4^{-2} = \dfrac{1}{16}$
4. $5^{-3} = \dfrac{1}{125}$ Group 2
5. $\left(\dfrac{1}{3}\right)^1 = \dfrac{1}{3}$ $\log_{\frac{1}{3}} \frac{1}{3} = 1$
6. $\left(\dfrac{1}{2}\right)^{-4} = 16$
7. $32^{4/5} = 16$
8. $81^{3/4} = 27$ $\log_{81} 27 = \frac{3}{4}$
9. $10^{-3} = 0.001$
10. $16^{-1/4} = 0.5$

In Exercises 11–16, given that $\log 3 \approx 0.4771$ and $\log 4 \approx 0.6021$, find the value of each logarithm.

11. $\log 12$
12. $\log \dfrac{3}{4}$
13. $\log 16$
14. $\log \sqrt{3}$
15. $\log 48$
16. $\log \dfrac{1}{300}$

In Exercises 17–20, write the expression as the logarithm of a single quantity.

17. $2 \ln a + 3 \ln b$
18. $\dfrac{1}{2} \ln x + 2 \ln y - 3 \ln z$
19. $\ln 3 + \dfrac{1}{2} \ln x + \ln y - \dfrac{1}{3} \ln z$ $x^{1/2} = \sqrt{x}$ $\ln \dfrac{3\sqrt{x}\,y}{\sqrt[3]{z}}$
20. $\ln 2 + \dfrac{1}{2} \ln(x + 1) - 2 \ln(1 + \sqrt{x})$ $\dfrac{2\sqrt{x+1}}{(1+\sqrt{x})^2}$

In Exercises 21–28, use the laws of logarithms to expand and simplify the expression.

21. $\log x(x + 1)^4$ **22.** $\log x(x^2 + 1)^{-1/2}$

23. $\log \dfrac{\sqrt{x+1}}{x^2+1}$ **24.** $\ln \dfrac{e^x}{1 + e^x}$

25. $\ln xe^{-x^2}$ **26.** $\ln x(x + 1)(x + 2)$

27. $\ln \dfrac{x^{1/2}}{x^2\sqrt{1 + x^2}}$ **28.** $\ln \dfrac{x^2}{\sqrt{x}(1 + x)^2}$

Group 3

need calculators

In Exercises 29–32, sketch the graph of the equation.

29. $y = \log_3 x$ **30.** $y = \log_{1/3} x$

31. $y = \ln 2x$ **32.** $y = \ln \dfrac{1}{2}x$

In Exercises 33 and 34, sketch the graphs of the equations on the same coordinate axes.

33. $y = 2^x$ and $y = \log_2 x$ **34.** $y = e^{3x}$ and $y = \dfrac{1}{3}\ln x$

In Exercises 35–44, use logarithms to solve the equation for t.

hint Take ln of each side

35. $e^{0.4t} = 8$ **36.** $\dfrac{1}{3}e^{-3t} = 0.9$

37. $5e^{-2t} = 6$ **38.** $4e^{t-1} = 4$

39. $2e^{-0.2t} - 4 = 6$ **40.** $12 - e^{0.4t} = 3$

41. $\dfrac{50}{1 + 4e^{0.2t}} = 20$ **42.** $\dfrac{200}{1 + 3e^{-0.3t}} = 100$

43. $A = Be^{-t/2}$ **44.** $\dfrac{A}{1 + Be^{t/2}} = C$

$e^t = 3$
$\ln e^t = \ln 3$
$t \ln e = \ln 3$
$t = \ln 3$
$t \approx 1.0986$

45. A function f has the form $f(x) = a + b \ln x$. Find f if it is known that $f(1) = 2$ and $f(2) = 4$.

go to 5.1 with group 1

46. AVERAGE LIFE SPAN One reason for the increase in human life span over the years has been the advances in medical technology. The average life span for American women from 1907 through 2007 is given by

$$W(t) = 49.9 + 17.1 \ln t \qquad (1 \le t \le 6)$$

where $W(t)$ is measured in years and t is measured in 20-year intervals, with $t = 1$ corresponding to 1907.

a. What was the average life expectancy for women in 1907?
b. If the trend continues, what will be the average life expectancy for women in 2027 ($t = 7$)?

Source: American Association of Retired Persons (AARP).

47. BLOOD PRESSURE A normal child's systolic blood pressure may be approximated by the function

$$p(x) = m(\ln x) + b$$

where $p(x)$ is measured in millimeters of mercury, x is measured in pounds, and m and b are constants. Given that $m = 19.4$ and $b = 18$, determine the systolic blood pressure of a child who weighs 92 lb.

48. MAGNITUDE OF EARTHQUAKES On the Richter scale, the magnitude R of an earthquake is given by the formula

$$R = \log \frac{I}{I_0}$$

where I is the intensity of the earthquake being measured and I_0 is the standard reference intensity.

a. Express the intensity I of an earthquake of magnitude $R = 5$ in terms of the standard intensity I_0.
b. Express the intensity I of an earthquake of magnitude $R = 8$ in terms of the standard intensity I_0. How many times greater is the intensity of an earthquake of magnitude 8 than one of magnitude 5?
c. In modern times, the greatest loss of life attributable to an earthquake occurred in Haiti in 2010. Known as the Haiti earthquake, it registered 7.0 on the Richter scale. How does the intensity of this earthquake compare with the intensity of an earthquake of magnitude $R = 5$?

49. SOUND INTENSITY The relative loudness of a sound D of intensity I is measured in decibels (db), where

$$D = 10 \log \frac{I}{I_0}$$

and I_0 is the standard threshold of audibility.

a. Express the intensity I of a 30-db sound (the sound level of normal conversation) in terms of I_0.
b. Determine how many times greater the intensity of an 80-db sound (rock music) is than that of a 30-db sound.
c. Prolonged noise above 150 db causes permanent deafness. How does the intensity of a 150-db sound compare with the intensity of an 80-db sound?

50. BAROMETRIC PRESSURE Halley's Law states that the barometric pressure (in inches of mercury) at an altitude of x mi above sea level is approximated by the equation

$$p(x) = 29.92e^{-0.2x} \qquad (x \ge 0)$$

If the barometric pressure as measured by a hot-air balloonist is 20 in. of mercury, what is the balloonist's altitude?

51. NEWTON'S LAW OF COOLING The temperature of a cup of coffee t min after it is poured is given by

$$T = 70 + 100e^{-0.0446t}$$

where T is measured in degrees Fahrenheit.

a. What was the temperature of the coffee when it was poured?
b. When will the coffee be cool enough to drink (say, 120°F)?