Properties Relating the Exponential and Logarithmic Functions

We made use of the relationship that exists between the exponential function $f(x) = e^x$ and the logarithmic function $g(x) = \ln x$ when we sketched the graph of g in Example 6. This relationship is further described by the following properties, which are an immediate consequence of the definition of the logarithm of a number.

Properties Relating e^x and $\ln x$

$$e^{\ln x} = x \qquad \text{(for } x > 0\text{)}$$

$$\ln e^x = x \qquad \text{(for any real number } x\text{)}$$

The to verify these properties.)

Properties 2 and 3, we conclude that the composite function satisfies

$$(f \circ g)(x) = f[g(x)]$$

$$= e^{\ln x} = x \qquad \text{(for all } x > 0\text{)}$$

$$(g \circ f)(x) = g[f(x)]$$

$$= \ln e^x = x \qquad \text{(for all } x > 0\text{)}$$

that satisfy this relationship are said to be inverses of each What the function f undoes what the function g does, and vice versa, so the The two functions in any order results in the identity function F(x) = x.* Equations (2) and (3) are useful in solving equathe avoice exponentials and logarithms.

TECHNOLOGY

Properties 2 and 3, which state that the and the logarithmic function $g(x) = \ln x$ are of the bottom as follows:

 $(x) = e^{\ln x}$, using the viewing window L 10 , Interpret the result.

of the equation by 2 to obtain

$$=\frac{5}{2}=2.5$$

the equation and using Equation

EXAMPLE 8 Solve the equation $5 \ln x + 3 = 0$.

Solution Adding -3 to both sides of the equation leads to

$$5 \ln x = -3$$

$$\ln x = -\frac{3}{5} = -0.6$$

and so

$$e^{\ln x} = e^{-0.6}$$

Using Equation (2), we conclude that

$$x = e^{-0.6}$$
$$\approx 0.55$$

Self-Check Exercises

- 1. Sketch the graph of $y = 3^x$ and $y = \log_3 x$ on the same set of axes.
- 2. Solve the equation $3e^{x+1} 2 = 4$.

Solutions to Self-Check Exercises 5.2 can be found on page 354.

Concept Questions

- 1. a. Define $y = \log_b x$.
- **b.** Define the logarithmic function f with base b. What restrictions, if any, are placed on b?
- 2. For the logarithmic function $y = \log_b x$ $(b > 0, b \ne 1)$, state (a) its domain and range, (b) its x-intercept, (c) where it is continuous, and (d) where it is increasing and where it is decreasing for the case b > 1 and the case b < 1.
- **3. a.** If x > 0, what is $e^{\ln x}$? **b.** If x is any real number, what is $\ln e^x$?
- **4.** Let $f(x) = \ln x^2$ and $g(x) = 2 \ln x$. Are f and g identical? Hint: Look at their domains.

Exercises

In Exercises 1–10, express each equation in logarithmic form.

$$1.2^6 = 64$$

2.
$$3^5 = 243$$
 $\log \left(\frac{1}{128}\right)$

$$3. 4^{-2} = \frac{1}{16}$$

4.
$$5^{-3} = \frac{1}{125}$$

5.
$$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$$
 / $\log \frac{1}{3}$ 6. $\left(\frac{1}{2}\right)^{-4} = 16$

$$6\left(\frac{1}{2}\right)^{-4} - 16$$

9.
$$10^{-3} = 0.001$$

8.
$$81^{3/4} = 27$$
 $\log_{10} 27 = 3$ **10.** $16^{-1/4} = 0.5$

In Exercises 11–16, given that log 3 \approx 0.4771 and log 4 \approx 0.6021, find the value of each logarithm.

12.
$$\log \frac{3}{4}$$

- **13.** log 16
- 16. $\log \frac{1}{300}$

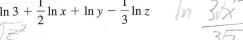
14. $\log \sqrt{3}$

In Exercises 17-20, write the expression as the logarithm of a single quantity.

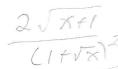
17. $2 \ln a + 3 \ln b$

$$18. \frac{1}{2} \ln x + 2 \ln y - 3 \ln z$$

19.
$$\ln 3 + \frac{1}{2} \ln x + \ln y - \frac{1}{3} \ln x$$



20.
$$\ln 2 + \frac{1}{2} \ln(x+1) - 2 \ln(1+\sqrt{x})$$



In Exercises 21–28, use the laws of logarithms to expand and simplify the expression.

21. $\log x(x+1)^4$ 22. $\log x(x^2+1)^{-1/2}$ 24. $\ln \frac{e^x}{1+e^x}$

21.
$$\log x(x+1)^4$$

22.
$$\log x(x^2+1)^{-1/2}$$

23.
$$\log \frac{\sqrt{x+1}}{x^2+1}$$

$$24. \ln \frac{e^x}{1+e^x} = \frac{1}{4}$$

25.
$$\ln xe^{-x^2}$$

26.
$$\ln x(x+1)(x+2)$$

27.
$$\ln \frac{x^{1/2}}{x^2 \sqrt{1+x^2}}$$

26.
$$\ln x(x+1)(x+2)$$
28. $\ln \frac{x^2}{\sqrt{x}(1+x)^2}$

In Exercises 29–32, sketch the graph of the equation.

29.
$$y = \log_3 x$$

30.
$$y = \log_{1/3} x$$

31.
$$y = \ln 2x$$

32.
$$y = \ln \frac{1}{2}x$$

In Exercises 33 and 34, sketch the graphs of the equations on the same coordinate axes.

33.
$$y = 2^x$$
 and $y = \log_2 x$

33.
$$y = 2^x$$
 and $y = \log_2 x$ **34.** $y = e^{3x}$ and $y = \frac{1}{3} \ln x$

In Exercises 35–44, use logarithms to solve the equation for t. 35. $e^{0.4t} = 8$ 36. $\frac{1}{3}e^{-3t} = 0.9$ 37. $5e^{-2t} = 6$ 39. $2e^{-0.2t} - 4 = 6$ 30. $12 - e^{0.4t} = 3$

35.
$$e^{0.4t} = 8$$

36.
$$\frac{1}{3}e^{-3t}=0.9$$

37.
$$5e^{-2t} = 6$$

38.
$$4e^{t-1} = 4$$

39.
$$2e^{-0.2t} - 4 = 6$$

40.
$$12 - e^{0.4t} = 3$$

41.
$$\frac{50}{1 + 4e^{0.2t}} = 20$$

42.
$$\frac{200}{1+3e^{-0.3t}}=100$$

43.
$$A = Be^{-t/2}$$

44.
$$\frac{A}{1 + Be^{t/2}} = C$$

- **45.** A function f has the form $f(x) = a + b \ln x$. Find f if it is known that f(1) = 2 and f(2) = 4.
- 46. Average Life Span One reason for the increase in human life span over the years has been the advances in medical technology. The average life span for American women from 1907 through 2007 is given by

$$W(t) = 49.9 + 17.1 \ln t$$
 $(1 \le t \le 6)$

where W(t) is measured in years and t is measured in 20-year intervals, with t = 1 corresponding to 1907.

- a. What was the average life expectancy for women in 1907?
- b. If the trend continues, what will be the average life expectancy for women in 2027 (t = 7)?

Source: American Association of Retired Persons (AARP).

47. BLOOD PRESSURE A normal child's systolic blood pressure may be approximated by the function

$$p(x) = m(\ln x) + b$$

where p(x) is measured in millimeters of mercury, x is measured in pounds, and m and b are constants. Given

that m = 19.4 and b = 18, determine the systolic blood pressure of a child who weighs 92 lb.

48. MAGNITUDE OF EARTHQUAKES On the Richter scale, the magnitude R of an earthquake is given by the formula

$$R = \log \frac{I}{I_0}$$

where I is the intensity of the earthquake being measured and I_0 is the standard reference intensity.

- **a.** Express the intensity I of an earthquake of magnitude R = 5 in terms of the standard intensity I_0 .
- **b.** Express the intensity I of an earthquake of magnitude R=8 in terms of the standard intensity I_0 . How many times greater is the intensity of an earthquake of magnitude 8 than one of magnitude 5?
- In modern times, the greatest loss of life attributable to an earthquake occurred in Haiti in 2010. Known as the Haiti earthquake, it registered 7.0 on the Richter scale. How does the intensity of this earthquake compare with the intensity of an earthquake of magnitude
- 49. Sound Intensity The relative loudness of a sound D of intensity I is measured in decibels (db), where

$$D = 10 \log \frac{I}{I_0}$$

- **a.** Express the intensity *I* of a 30-db sound (the sound level of normal conversation) in terms of I_0 .
- 42. $\frac{200}{1+3e^{-0.3t}} = 100$ 3 and I_0 is the standard threshold of audibility. level of normalization and I_0 is the standard threshold of audibility. b. Determine how many times greater the intensity of an 80-db sound (rock music) is than that of a 30-db sound.
 - c. Prolonged noise above 150 db causes permanent deafness. How does the intensity of a 150-db sound compare with the intensity of an 80-db sound?
 - 50. BAROMETRIC PRESSURE Halley's Law states that the barometric pressure (in inches of mercury) at an altitude of x mi above sea level is approximated by the equation

$$p(x) = 29.92e^{-0.2x} \qquad (x \ge 0)$$

If the barometric pressure as measured by a hot-air balloonist is 20 in. of mercury, what is the balloonist's altitude?

51. Newton's Law of Cooling The temperature of a cup of coffee t min after it is poured is given by

$$T = 70 + 100e^{-0.0446t}$$

where T is measured in degrees Fahrenheit.

- a. What was the temperature of the coffee when it was
- b. When will the coffee be cool enough to drink (say, 120°F)?