Chapter 3.2 – The Product and Quotient Rules

Rule 5: The Product Rule

$$\frac{d}{dx}[f(x)\cdot g(x)] = g(x)\cdot f'(x) + f(x)\cdot g'(x)$$

EXAMPLE 1:

$$f(x) = (2x^3 - 5)(4x + 9)$$

Rule 6: The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2},\ g(x) \neq 0$$

EXAMPLE 2:

$$h(x) = \frac{x^2 + 1}{x^2 - 1}$$

Find the derivatives of the following. Simplify your answers.

1. $f(x) = (3x^2)(x-1)$

2. $f(x) = (x^3-1)(x+1)$

3. $f(x) = (x^3-12x)(3x^2+2x)$

4. $f(x) = \dfrac{x}{2x-4}$ [2 lines]

5. $f(x)=\dfrac{3}{2x+4}$ [2 lines]

6. $f(x)=\dfrac{x^2+x-2}{x^3+6}$ [2 lines]

7. $f(t)=\dfrac{t^2}{3t^2-2t+1}$ [2 lines]

Chapter 3.3 – The Chain Rule

So far, we have found the derivative of single terms with powers/exponents. The Power Rule was sufficient in those cases. Now we will consider finding the derivative of functions with powers. For example:

$$g(x) = (x^3 - 2x^2 + 6x - 5)^4$$

A simple but tedious solution would be to simplify the function by getting rid of the parentheses by multiplying the polynomial four times and then using the power rule. Instead, we will use what is known as the Chain Rule. The Chain Rule is as follows:

$$\begin{aligned} &If \quad h(x) = [f(x)]^n, \\ &then\ h'(x) = n[f(x)]^{n-1} \cdot f'(x) \end{aligned}$$

Think of the $[\quad]^n$ as the "outside" of $h(x)$ and $f(x)$ as the inside of $h(x)$. So the Chain Rule finds the derivative $h'(x)$ by first finding the derivative of the outside $n[\quad]^{n-1}$ and multiplying it by the derivative of the inside $f'(x)$.

Looking again at the function, $g(x) = (x^3 - 2x^2 + 6x - 5)^4$

$$g'(x) = \underbrace{4(x^3 - 2x^2 + 6x - 5)^{4-1}}_{outside\ \ derivative} \cdot \underbrace{(x^3 - 2x^2 + 6x - 5)'}_{inside\ \ derivative}$$

$$g'(x) = 4(x^3 - 2x^2 + 6x - 5)^3 \cdot (3x^2 - 4x + 6)$$

Q: When do you use the Power Rule versus the Chain Rule?

A: The Power Rule is used for single variables raised to powers. The Chain Rule is used when there is a "chain" of terms raised to a power – think of the "+" and "-" as the *links* of the chain connecting the terms.

Find the derivative of the following using the chain rule: (Note that the derivative may also involve the product and quotient rules).

1. $f(x) = (x^2 + 2)^5$ [2 lines]

2. $f(x) = (2x + 1)^{-2}$ [2 lines]

3. $f(x) = (1 + x^2)^5 \cdot (1 - 2x^2)^8$ [2 lines]

4. $f(x) = \left(\dfrac{x+1}{x-1} \right)^5$ [2 lines]

Chapter 3.5 – Higher-Order Derivatives

Higher-order derivatives are found by taking the derivative of a derivative. For example, the second derivative of a function is the derivative of the first derivative. The notation is as follows:

$$f''(x) = [f'(x)]', \quad where\ f(x)\, is\, a\ function$$

Any derivative higher than the third $f'''(x)$ is denoted by the number itself, for example $f^{(4)}, f^{(5)}, f^{(6)}$ etc.

The relevance of higher order derivatives can be seen with distance, velocity and acceleration.

Velocity, v = $\dfrac{change\ \ in\ dis\tan ce}{change\ in\ time}$ = the derivative of the distance formula, s

$$v = s'$$

Acceleration, a = $\dfrac{change\ in\ velocity}{change\ in\ time}$ = the derivative of the velocity formula, v

$$a = v' = (s')' = s''$$

EXAMPLE 1:

Find up to the sixth derivative of the function $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$

$f'(x) =$

$f''(x) =$

$f'''(x) =$

$f^{(4)}(x) =$

$f^{(5)}(x) =$

$f^{(6)}(x) =$

Find the second derivative of the following:

1. $f(x) = -0.2x^2 + 0.3x + 4$

2. $g(t) = -3t^3 + 24t^2 + 6t - 64$

3. $f(x) = x^5 - 3x^4 + 4x^3 - 2x^2 + 9$

4. The distance, s, covered by a car after t seconds is given by

   $s = -t^3 + 8t^2 + 20t,\ 0 \le t \le 6$.

   Find the general expression for the velocity of the car and the acceleration of the car at time t.