Chapter 3.6 – Implicit Differentiation and Related Rates

The function $y = f(x)$ is said to be in *explicit* form as y is explicitly expressed in terms of x. The "y" is isolated on the left of the equal sign and is considered to be on the "outside" of the function. (Think **EX**ternal = **EX**plicit). The following is an example of an explicit equation:

$$y = \frac{x^2}{1-x}$$

Consider the example:

$$x^2 y + 1 - x^2 + y = 0$$

This equation is in *implicit* form. That is, the "y" is mixed up inside of the equation. (Think **In**ternal = **Im**plicit).

IMPLICIT DIFFERENTIATION

Implicit differentiation is done when equations are given in implicit form, that is, when "y" is not isolated to one side of the equal sign and uses the notation $\frac{dy}{dx}$.

$\frac{dy}{dx}$ is another way to represent the derivative. One interpretation of $\frac{dy}{dx}$ is "the change in y with respect to x". In other words, how y is changing with respect to how x is changing. Previously the derivative y' was written in terms of "x". Here "x" is seen as the "important" variable. With implicit differentiation, x will still be seen as the important variable – the one you find the derivative of.

EXAMPLE 2:

$$x^2 + y^2 = 6$$

Notice that "y" is not isolated on one side of the equal sign. To find the derivative, we will find the derivative of the first term:

$$(x^2)' = 2x$$

Now we will find the derivative of the second term. Since it is not an "x", the important variable, we must indicate this.

$$(y^2)' = 2y \cdot \frac{dy}{dx}$$

The dy tells us which (unimportant) variable we are differentiating and the dx tells us which variable is the important one. Technically the 2x should also have a dx/dx next to it, but these terms cancel each other out.

Lastly, we do the derivative of the constant.

$$(6)' = 0$$

The constant does not have a variable so nothing more is needed.

The final answer is: $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

<u>EXAMPLE 3</u>

Find the derivative of $3x^4 + x^3y + 7y = 2$

Find the derivative:

1. $x^3 + y^3 + y - 4 = 0$

2. $x^2 y^3 - 2xy^2 = 5$

Related Rates:

Implicit differentiation is essential for solving related rates questions. Related rates questions have one thing in common – they describe **the rate** at which a quantity is changing (usually with respect to time). Speed/velocity is an example of a related rate. It describes the change in distance with respect to the change in time:

$$Velocity, v = \frac{\Delta dis\tan ce}{\Delta time} = \frac{\Delta s}{\Delta t}$$

Note that this translates into a derivative s' as well as an example of implicit differentiation (the derivative of "s" when "t" is the important variable. In the examples we will look at, the derivative with respect to time (a related rate) will be fundamental in our solutions.

Steps for solving related-rates problems:

a. Assign variables for the quantities mentioned. (A picture/diagram is often useful).
b. Note any quantities and/or related rates given and the quantity to be found.
c. Write an equation for the relationship between variables.
d. Differentiate both sides of the equation with respect to time, t.
e. Solve for the unknown quantity/quantity to be found.

EXAMPLE 3:

A small hole is made in the bottom of a paint can. As the paint leaks out, it forms a circle on the ground. The rate at which the radius of the circle is increasing is 2 cm/min. When the radius of the paint circle is 40cm, how fast is the area of the circle increasing?

a.

b.

c.

d.

e.

Solve the following related rates:

3. A spherical ball of snow is melting at a rate of $72\pi \ ft^3 / \sec$ (its volume is decreasing at that rate). At the moment that its diameter is 6ft, how fast is the radius decreasing?

4. A 40-foot ladder that is leaning against a wall begins to slide. How fast is the bottom of the ladder sliding away from the wall at the time when the top of the ladder is sliding down the wall at a rate of 9 ft/sec and the bottom of the ladder is 24 feet from the wall? [Hint: Use Pythagoras' Theorem to form a relationship between variables and consider movement down the wall to be negative.]

5. Air is being pumped into a spherical balloon at a rate of 5 cubic centimetres per minute. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20cm.

6. A 17-ft ladder (much more realistic) is resting against a wall. Jake is balancing precariously at the very top of the ladder. The bottom of the ladder is 8 feet away from the wall, so you decide to kick the ladder away from the house at a rate of ¼ ft/sec. How fast is Jake falling? Why are you and Jake no longer friends?

7. A screen saver displays the outline of a 3cm by 2cm rectangle and then expands the rectangle in such a way that the 2cm side is expanding at a rate of 4cm/sec and the proportions of the rectangle never change. How fast is the area of the rectangle increasing when its dimensions are 12cm by 8cm?