22. We want to maximize the profit function $P(x,y) = 4x + 2y$ subject to the constraint $g(x,y) = 2x^2 + y - 3 = 0$. The Lagrangian function is $F(x,y,\lambda) = P(x,y) + \lambda g(xy) = 4x + 2y + \lambda(2x^2 + y - 3)$. To find the critical point of F, we solve the system $\begin{cases} F_x = 4 + 4\lambda x = 0 \\ F_y = 2 + \lambda = 0 \\ F_\lambda = 2x^2 + y - 3 = 0 \end{cases}$ Solving the second equation yields $\lambda = -2$. Substituting this value into the first equation, we obtain $x = -\frac{4}{4(-2)} = \frac{1}{2}$. Substituting this value of x into the third equation in the system, we have $y = 3 - 2\left(\frac{1}{2}\right)^2 = \frac{5}{2}$. Thus, the company should produce 500 type A and 2500 type B souvenirs per week.

23. Let the dimensions of the box (in feet) be $x \times y \times z$. We want to maximize $f(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = 2x + y + 2z - 108 = 0$. The Lagrangian function is $F(x,y,\lambda) = xyz + \lambda(2x + y + 2z - 108)$. To find the critical points of F, we solve the system $\begin{cases} F_x = yz + 2\lambda = 0 & (1) \\ F_y = xz + \lambda = 0 & (2) \\ F_z = xy + 2\lambda = 0 & (3) \\ F_\lambda = 2x + y + 2z - 108 = 0 & (4) \end{cases}$ Multiplying (1) by x and (2) by y, we obtain $\begin{cases} xyz + 2\lambda x = 0 & (5) \\ xyz + \lambda y = 0 & (6) \end{cases}$ Subtracting (5) from (6), we have $\lambda(y - 2x) = 0$. Since $\lambda \neq 0$, we see that $y = 2x$. Next, multiplying (3) by z, we obtain $xyz + 2\lambda z = 0$ (7). Subtracting (5) from (7) gives $2\lambda(z - x) = 0$, so $z = x$, and substituting $y = 2x$ and $z = x$ into (4) gives $2x + 2x + 2x = 108$, so $6x = 108$, giving $x = 18$. Thus, $y = 2(18) = 36$ and $z = 18$, and so the required dimensions are $18'' \times 18'' \times 36''$.

24. $V = \pi r^2 \ell$. The constraint is $2\pi r + \ell = 130$, so $g(r,\ell) = 2\pi r + \ell - 130$. The Lagrangian function is $F(r,\ell,\lambda) = \pi r^2 \ell + \lambda(2\pi + \ell - 130)$, so we solve the system $\begin{cases} F_r = 2\pi r\ell + 2\pi\lambda = 0 \\ F_\ell = \pi r^2 + \lambda = 0 \\ F_\lambda = 2\pi r + \ell - 130 = 0 \end{cases}$

The second equation gives $\lambda = -\pi r^2$. Substituting into the first equation gives $2\pi r\ell + 2\pi(-\pi r^2) = 0$, so $2\pi r(\ell - \pi r) = 0$. Because $r \neq 0$, we have $\ell = \pi r$, which we substitute into the third equation to obtain $2\pi r + \pi r - 130 = 0$, $3\pi r = 130$, and so $r = \frac{130}{3\pi}$. Therefore, $\ell = \pi\left(\frac{130}{3\pi}\right) = \frac{130}{3}$, or $43\frac{1}{3}''$. The volume is $\pi r^2 \ell = \pi\left(\frac{130}{3\pi}\right)^2\left(\frac{130}{3}\right) = \frac{2{,}197{,}000}{27\pi}$, or $\frac{2{,}197{,}000}{27\pi}$ in^3.

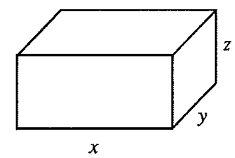
25. We want to minimize the function $C(r,h) = 8\pi rh + 6\pi r^2$ subject to the constraint $\pi r^2 h - 64 = 0$. We form the Lagrangian function $F(r,h,\lambda) = 8\pi rh + 6\pi r^2 - \lambda(\pi r^2 h - 64)$ and solve the system

$$\begin{cases} F_r = 8\pi h + 12\pi r - 2\lambda\pi rh = 0 \\ F_h = 8\pi r - \lambda\pi r^2 = 0 \\ F_\lambda = \pi r^2 h - 64 = 0 \end{cases}$$

Solving the second equation for λ yields $\lambda = 8/r$, which when substituted into the first equation yields $8\pi h + 12\pi r - 2\pi rh\left(\frac{8}{r}\right) = 0$, $12\pi r = 8\pi h$, and $h = \frac{3}{2}r$. Substituting this value of h into the third equation of the system, we find $3r^2\left(\frac{3}{2}r\right) = 64$, $r^3 = \frac{128}{3\pi}$, so $r = \frac{4}{3}\sqrt[3]{\frac{18}{\pi}}$ and $h = 2\sqrt[3]{\frac{18}{\pi}}$.

26. We form the Lagrangian function $F(x, y, \lambda) = xyz + \lambda(3xy + 2xz + 2yz - 36)$ and solve the system

$$\begin{cases} F_x = yz + 3\lambda y + 2\lambda z = 0 \\ F_y = xz + 3\lambda x + 2\lambda z = 0 \\ F_z = xy + 2\lambda x + 2\lambda y = 0 \\ F_\lambda = 3xy + 2xz + 2yz - 36 = 0 \end{cases}$$



Multiplying the first, second, and third equations by x, y, and z respectively, we obtain

$$\begin{cases} xyz + 3\lambda xy + 2\lambda xz = 0 \\ xyz + 3\lambda xy + 2\lambda yz = 0 \\ xyz + 2\lambda xz + 2\lambda yz = 0 \end{cases}$$

Subtracting the second equation from the first and the third equation from the second yields $\begin{cases} 2\lambda(x - y)z = 0 \\ \lambda x(3y - 2z) = 0 \end{cases}$

Solving this system, we find that $x = y$ and $x = \frac{3}{2}y$. Substituting these values into the third equation, we find that $3y^2 + 2y\left(\frac{3}{2}\right)y + 2y\left(\frac{3}{2}\right) - 36 = 0$, and so $y = \pm 2$. We reject the negative root find that $x = 2$, $y = 2$, and $z = 3$ provides the desired relative maximum. Thus, the dimensions are $2' \times 2' \times 3'$.

27. Let the box have dimensions x by y by z feet. Then $xyz = 4$. We want to minimize $C = 2xz + 2yz + \frac{3}{2}(2xy) = 2xz + 2yz + 3xy$. We form the Lagrangian function $F(x, y, z, \lambda) = 2xz + 2yz + 3xy + \lambda(xyz - 4)$ and solve the system

$$\begin{cases} F_x = 2z + 3y + \lambda yz = 0 \\ F_y = 2z + 3x + \lambda xz = 0 \\ F_z = 2x + 2y + \lambda xy = 0 \\ F_\lambda = xyz - 4 = 0 \end{cases}$$

Multiplying the first, second, and third equations by x, y, and z respectively, we have

$$\begin{cases} 2xz + 3xy + \lambda xyz = 0 \\ 2yz + 3xy + \lambda xyz = 0 \\ 2xz + 2yz + \lambda xyz = 0 \end{cases}$$

The first two equations imply that $2z(x - y) = 0$. Because $z \neq 0$, we see that $x = y$. The second and third equations imply that $x(3y - 2z) = 0$ or $x = \frac{3}{2}y$. Substituting these values into the fourth equation in the system, we find $y^2\left(\frac{3}{2}y\right) = 4$, so $y^3 = \frac{8}{3}$. Therefore, $y = \frac{2}{3^{1/3}} = \frac{2}{3}\sqrt[3]{9}$, $x = \frac{2}{3}\sqrt[3]{9}$, and $z = \sqrt[3]{9}$, and the dimensions (in feet) are $\frac{2}{3}\sqrt[3]{9} \times \frac{2}{3}\sqrt[3]{9} \times \sqrt[3]{9}$.

28. Let x, y, and z denote the length, width, and height of the box. We can assume without loss of generality that the cost of the material for constructing the sides is $\$1/\text{ft}^2$. Then the total cost is $C = f(x, y, z) = 3xy + 2xz + 2yz$. We want to minimize f subject to the constraint $g(x, y, z) = xyz - 12 = 0$. We form the Lagrangian

$F(x, y, z, \lambda) = 3xy + 2xz + 2yz - \lambda(xyz - 12)$ and solve the system $\begin{cases} F_x = 3y + 2z - \lambda yz = 0 \\ F_y = 3x + 2z - \lambda xz = 0 \\ F_z = 2x + 2y - \lambda xy = 0 \\ F_\lambda = xyz - 12 = 0 \end{cases}$

From the first and second equations, we find $\lambda = \dfrac{3y + 2z}{yz} = \dfrac{3x + 2z}{xz} \Rightarrow 3xyz + 2xz^2 = 3xyz + 2yz^2 \Rightarrow x = y$.

From the second and third equations, we have $\lambda = \dfrac{3x + 2z}{xz} = \dfrac{2x + 2y}{xy} \Rightarrow 3x^2y + 2xyz = 2x^2z + 2xyz \Rightarrow$ $z = \frac{3}{2}y$. Substituting into the fourth equation, we have $y(y)\left(\frac{3}{2}y\right) = 12 \Rightarrow y^3 = 8 \Rightarrow y = 2$. Thus, $x = 2$ and $z = 3$. The dimensions of the box are $2' \times 2' \times 3'$.

29. Let x, y, and z denote the length, width, and height of the box. We can assume without loss of generality that the cost of the material for constructing the sides and top is $\$1/\text{ft}^2$. Then the total cost is $C = f(x, y, z) = 3xy + 2xz + 2yz$. We want to minimize f subject to the constraint $g(x, y, z) = xyz - 16 = 0$. We form the Lagrangian

$F(x, y, z, \lambda) = 3xy + 2xz + 2yz - \lambda(xyz - 16)$ and solve the system $\begin{cases} F_x = 3y + 2z - \lambda yz = 0 \\ F_y = 3x + 2z - \lambda xz = 0 \\ F_z = 2x + 3y - \lambda xy = 0 \\ F_\lambda = xyz - 16 = 0 \end{cases}$

From the first and second equations, we find $\lambda = \dfrac{3y + 2z}{yz} = \dfrac{3x + 2z}{xz} \Rightarrow 3xyz + 2xz^2 = 3xyz + 2yz^2 \Rightarrow x = y$.

From the second and third equations, we have $\lambda = \dfrac{3x + 2z}{xz} = \dfrac{2x + 2y}{xy} \Rightarrow 3x^2y + 2xyz = 2x^2z + 2xyz \Rightarrow$ $z = \frac{3}{2}y$. Substituting into the fourth equation, we have $y(y)\left(\frac{3}{2}y\right) = 16 \Rightarrow y^3 = \frac{32}{3} \Rightarrow y = \frac{2}{3}\sqrt[3]{36}$. Thus, $x = \frac{2}{3}\sqrt[3]{36}$ and $z = \sqrt[3]{36}$. The dimensions of the box are $\frac{2}{3}\sqrt[3]{36}' \times \frac{2}{3}\sqrt[3]{36}' \times \sqrt[3]{36}'$.

30. We want to maximize $f(x, y) = 90x^{1/4}y^{3/4}$ subject to $x + y = 60{,}000$. We form the Lagrangian function $F(x, y, \lambda) = 90x^{1/4}y^{3/4} + \lambda(x + y - 60{,}000)$ and solve the system

$\begin{cases} F_x = \frac{45}{2}x^{-3/4}y^{3/4} + \lambda = 0 \\ F_y = \frac{135}{2}x^{1/4}y^{-1/4} + \lambda = 0 \\ F_\lambda = x + y - 60{,}000 = 0 \end{cases}$ Eliminating λ in the first two equations gives $\dfrac{45}{2}\left(\dfrac{y}{x}\right)^{3/4} - \dfrac{135}{2}\left(\dfrac{x}{y}\right)^{1/4} = 0$,

so $\dfrac{y}{x} - 3 = 0$, and $y = 3x$. Substituting this value into the third equation in the system, we find $x + 3x = 60{,}000$, so $x = 15{,}000$, and $y = 45{,}000$. Thus, the company should spend \$15,000 on newspaper advertisements and \$45,000 on television advertisements.

31. We want to maximize $P(x,y)$ subject to $g(x,y) = px + qy - C = 0$. First, we form the Lagrangian $F(x,y) = P(x,y) + \lambda(px + qy - C)$. Next, we solve the system $F_x = P_x + \lambda p = 0$, $F_y = Py + \lambda q = 0$, $F_\lambda = px + qy - C = 0$. From the first two equations, we see that $\lambda = -\frac{P_x}{p} = -\frac{P_y}{q}$, so if (x^*, y^*) gives rise to a relative maximum value of P subject to the constraint $g(x,y) = 0$, then $\frac{P_x(x^*,y^*)}{p} = \frac{P_y(x^*,y^*)}{q}$ or $\frac{P_x(x^*,y^*)}{P_y(x^*,y^*)} = \frac{p}{q}$.

32. Using the result of Exercise 31, we have $\frac{f_x(x^*,y^*)}{f_y(x^*,y^*)} = \frac{p}{q}$. But $f_x(x,y) = abx^{b-1}y^{1-b}$ and $f_y(x,y) = a(1-b)x^by^{-b}$, so $\frac{f_x(x,y)}{f_y(x,y)} = \frac{abx^{b-1}y^{1-b}}{a(1-b)x^by^{-b}} = \frac{bx^{-1}y}{1-b} = \frac{by}{(1-b)x}$. At the maximum level of production, $\frac{by^*}{(1-b)x^*} = \frac{p}{q}$ or $y^* = \frac{(1-b)px^*}{bq}$. Substituting this into the equation $px^* + qy^* = C$ gives $px^* + \frac{(1-b)pq}{bq}x^* = C$, whence $\left[p + \frac{(1-b)p}{b}\right]x^* = C$, so $x^* = \frac{bC}{p}$ and thus $y^* = \frac{(1-b)p}{bq} \cdot \frac{bC}{p} = \frac{(1-b)C}{q}$. Thus, the amounts to be spent on labor and capital are $\frac{bC}{p} \cdot p = bC$ and $\frac{(1-b)C}{q} \cdot q = (1-b)C$, respectively.

33. We want to maximize $f(x,y) = 100x^{3/4}y^{1/4}$ subject to $100x + 200y = 200{,}000$. We form the Lagrangian function $F(x,y,\lambda) = 100x^{3/4}y^{1/4} + \lambda(100x + 200y - 200{,}000)$ and solve the system

$$\begin{cases} F_x = 75x^{-1/4}y^{1/4} + 100\lambda = 0 \\ F_y = 25x^{3/4}y^{-3/4} + 200\lambda = 0 \\ F_\lambda = 100x + 200y - 200{,}000 = 0 \end{cases}$$

The first two equations imply that $150x^{-1/4}y^{1/4} - 25x^{3/4}y^{-3/4} = 0$ or, upon multiplying by $x^{1/4}y^{3/4}$, $150y - 25x = 0$, which implies that $x = 6y$. Substituting this value of x into the third equation of the system, we have $600y + 200y - 200{,}000 = 0$, giving $y = 250$, and therefore $x = 1500$. So to maximize production, he should buy 1500 units of labor and 250 units of capital.

34. We want to minimize $C = 2xy + 8xz + 6yz$ subject to $xyz = 12{,}000$. We form the Lagrangian function $F(x,y,z,\lambda) = 2xy + 8xz + 6yz + \lambda(xyz - 12{,}000)$ and solve the system

$$\begin{cases} F_x = 2y + 8z + \lambda yz = 0 \\ F_y = 2x + 6z + \lambda xz = 0 \\ F_z = 8x + 6y + \lambda xy = 0 \\ F_\lambda = xyz - 12{,}000 = 0 \end{cases}$$

Multiplying the first, second, and third equations by x, y, and z, we obtain

$$\begin{cases} 2xy + 8yz + \lambda xyz = 0 \\ 2xy + 6yz + \lambda xyz = 0 \\ 8xz + 6yz + \lambda xyz = 0 \end{cases}$$

The first two equations imply that $z(8x - 6y) = 0$ so, because $z \neq 0$, we have $x = \frac{3}{4}y$. The second and third equations imply that $x(8z - 2y) = 0$, so $x = \frac{1}{4}y$. Substituting these values into the third equation of the system, we have $\left(\frac{3}{4}y\right)(y)\left(\frac{1}{4}y\right) = 12{,}000$, so $y^3 = 64{,}000$ and $y = 40$. Therefore, $x = 30$ and $z = 10$. The heating cost is thus $C = 2(30)(40) + 8(30)(10) + 6(40)(10) = 7200$, or \$7200, as obtained earlier.

35. We use the result of Exercise 33 with $P(x,y) = f(x,y) = 100x^{3/4}y^{3/4}$, $P = 100$, $q = 200$, and $C = 200{,}000$. Here $P_x(x,y) = f_x(x,y) = 100\left(\frac{3}{4}x^{-1/4}y^{1/4}\right) = 75\left(\frac{y}{x}\right)^{1/4}$ and $P_y(x,y) = f_y(x,y) = 100\left(\frac{1}{4}x^{3/4}y^{-3/4}\right) = 25\left(\frac{x}{y}\right)^{3/4}$. Thus, $\dfrac{P_x(x,y)}{P_y(x,y)} = \dfrac{p}{q}$ gives $\dfrac{75\,(y/x)^{1/4}}{25\,(x/y)^{3/4}} = \dfrac{100}{200}$, $\dfrac{3y^{1/4}y^{3/4}}{x^{1/4}x^{3/4}} = \dfrac{1}{2}$, $\dfrac{y}{x} = \dfrac{1}{6}$, and $y = \frac{1}{6}x$. Substituting this into the constraint equation $100x + 200y = 200{,}000$ yields $100x + \frac{200}{6}x = 200{,}000$, $600x + 200x = 12{,}000{,}000$, and $x = 1500$, and so $y = \frac{1500}{6} = 250$. Therefore, 1500 units should be expended on labor and 250 units on capital, as obtained earlier.

36. We want to minimize $C(x,y) = px + qy$ subject to $P(x,y) = k$. The Lagrangian is $F(x,y) = px + qy + \lambda(P(x,y) - k)$. We solve the system

$$\begin{cases} F_x(x,y) = p + \lambda P_x(x,y) = 0 \\ F_y(x,y) = q + \lambda P_y(x,y) = 0 \\ F_\lambda(x,y) = P(x,y) - k = 0 \end{cases}$$

From the first two equations, we find $\lambda = -\dfrac{p}{P_x(x,y)} = -\dfrac{q}{P_y(x,y)}$, so if (x^*,y^*) gives a relative minimum value of C subject to $F_\lambda(x,y) = P(x,y) - k = 0$, then we must have $-\dfrac{p}{P_x(x^*,y^*)} = -\dfrac{q}{P_y(x^*,y^*)}$, and so $\dfrac{P_x(x^*,y^*)}{P_y(x^*,y^*)} = \dfrac{p}{q}$.

37. We use the result of Exercise 36 with $P(x,y) = f(x,y) = ax^by^{1-b}$. Here $P_x(x,y) = abx^{b-1}y^{1-b}$ and $P_y(x,y) = a(1-b)x^by^{-b}$, so $\dfrac{P_x(x,y)}{P_y(x,y)} = \dfrac{abx^{b-1}y^{1-b}}{a(1-b)x^by^{-b}} = \dfrac{bx^{-1}y}{1-b} = \dfrac{by}{(1-b)x}$. At the production level with minimum cost, we have $\dfrac{by}{(1-b)x} = \dfrac{p}{q}$, so $y = \dfrac{(1-b)px}{bq}$. Substituting this into the equation $ax^by^{1-b} = k$, we obtain $ax^b\left[\dfrac{(1-b)px}{bq}\right]^{1-b} = k$, whence $ax^b\left[\dfrac{(1-b)p}{bq}\right]^{1-b}x^{1-b} = k$, $ax\left[\dfrac{(1-b)p}{bq}\right]^{1-b} = k$, and $x = \dfrac{k}{a}\left[\dfrac{bq}{(1-b)p}\right]^{1-b}$, so $y = \dfrac{(1-b)p}{bq}\cdot\dfrac{k}{a}\left[\dfrac{bq}{(1-b)p}\right]^{1-b} = \dfrac{k}{a}\left[\dfrac{bq}{(1-b)p}\right]^{-b} = \dfrac{k}{a}\left[\dfrac{(1-b)p}{bq}\right]^{b}$. Thus, $x^*p = \dfrac{kp}{a}\left[\dfrac{bq}{(1-b)p}\right]^{1-b}$ should be spent on labor and $y^*q = \dfrac{kq}{a}\left[\dfrac{1-b}{bq}p\right]^b$ should be spent on capital.

38. We use the result of Exercise 37 with $a = 100$, $b = \frac{3}{4}$, $p = 100$, $q = 200$, and $k = 2000$, and find that

$$x^* = \frac{2000}{100}\left[\frac{\frac{3}{4}(200)}{\left(1-\frac{3}{4}\right)(100)}\right]^{1-(3/4)} = 20\left(\frac{150}{25}\right)^{1/4} = 20\,(6)^{1/4} \text{ and } y^* = \frac{2000}{100}\left(\frac{25}{150}\right)^{3/4} = 20\left(\tfrac{1}{6}\right)^{3/4}.$$

Thus, he should spend $100\,(20)\,(6)^{1/4} \approx 3130$ (dollars) on labor and $200\,(20)\left(\frac{1}{6}\right)^{3/4} \approx 1043$ (dollars) on capital.

39. False. See Example 1.

40. False. See Example 1.

41. True. We form the Lagrangian function $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$. Then $F_x = 0$, $F_y = 0$, and $F_\lambda = 0$ at (a,b) and $f_x(a,b) + \lambda g_x(a,b) = 0$, so $f_x(a,b) = -\lambda g_x(a,b)$, and $f_y(a,b) + \lambda(a,b) = 0$, so $f_y(a,b) = -\lambda g_y(a,b)$ and $g(a,b) = 0$.

42. True.

8.6 Double Integrals

Concept Questions page 630

1. An iterated integral is a single integral such as $\int_a^b f(x,y)\,dx$, where we think of y as a constant. It is evaluated as follows: $\int_R \int f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$.

2. $\int_R \int f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$.

3. $\int_R \int f(x,y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$.

4. It gives the volume of the solid region bounded above by the graph of f and below by the region R.

5. The average value is $\dfrac{\int_R \int f(x,y)\,dA}{\int_R \int dA}$.

Exercises page 631

1. $\int_1^2 \int_0^1 (y+2x)\,dy\,dx = \int_1^2 \left(\frac{1}{2}y^2+2xy\right)\Big|_{y=0}^{y=1} dx = \int_1^2 \left(\frac{1}{2}+2x\right) dx = \left(\frac{1}{2}x+x^2\right)\Big|_1^2 = 5-\frac{3}{2}=\frac{7}{2}$.

2. $\int_0^2 \int_{-1}^2 (x+2y)\,dx\,dy = \int_0^2 \left(\frac{1}{2}x^2+2xy\right)\Big|_{x=-1}^{x=2} dy = \int_0^2 \left[(2+4y)-\left(\frac{1}{2}-2y\right)\right] dy = \int_0^2 \left(\frac{3}{2}+6y\right) dy$

$$= \left(\frac{3}{2}y+3y^2\right)\Big|_0^2 = 3+12=15.$$

3. $\int_{-1}^1 \int_0^1 xy^2\,dy\,dx = \int_{-1}^1 \frac{1}{3}xy^3\Big|_{y=0}^{y=1} dx = \int_{-1}^1 \frac{1}{3}x\,dx = \frac{1}{6}x^2\Big|_{-1}^1 = \frac{1}{6}-\left(\frac{1}{6}\right)=0$.

4. $\int_0^1 \int_0^2 (12xy^2+8y^3)\,dy\,dx = \int_0^1 (4xy^3+2y^4)\big|_{y=0}^{y=2} dx = \int_0^1 (32x+32)\,dx = (16x^2+32x)\big|_0^1 = 48$.

5. $\displaystyle\int_{-1}^2 \int_1^{e^3} \frac{x}{y}\,dy\,dx = \int_{-1}^2 x\ln y\big|_{y=1}^{y=e^3} dx = \int_{-1}^2 x\ln e^3\,dx = \int_{-1}^2 3x\,dx = \frac{3}{2}x^2\Big|_{-1}^2 = \frac{3}{2}(4)-\frac{3}{2}(1)=\frac{9}{2}$.

6. $\displaystyle\int_0^1 \int_{-2}^2 \frac{xy}{1+y^2}\,dx\,dy = \int_0^1 \left[\frac{1}{2}\left(\frac{y}{1+y^2}\right)x^2\right]_{x=-2}^{x=2} dy = \int_0^1 0\,dy = 0$.

7. $\int_{-2}^0 \int_0^1 4xe^{2x^2+y}\,dx\,dy = \int_{-2}^0 e^{2x^2+y}\Big|_{x=0}^{x=1} dy = \int_{-2}^0 (e^{2+y}-e^y)\,dy = (e^{2+y}-e^y)\big|_{-2}^0 = (e^2-1)-(e^0-e^{-2})$

$$= e^2-2+e^{-2} = (e^2-1)(1-e^{-2}).$$

8. $\displaystyle\int_0^1 \int_1^2 \frac{y}{x^2}e^{y/x}\,dx\,dy = \int_0^1 (-e^{y/x})\big|_{x=1}^{x=2} dy = \int_0^1 (-e^{y/2}+e^y)\,dy = (-2e^{y/2}+e^y)\big|_0^1$

$$= (-2e^{1/2}+e) - (-2+1) = -2e^{1/2}+e+1.$$

9. $\int_0^1 \int_1^e \ln y\,dy\,dx = \int_0^1 (y\ln y - y)\big|_{y=1}^{y=e} dx = \int_0^1 dx = 1$.

10. $\int_1^e \int_1^{e^2} \frac{\ln y}{x}\,dx\,dy = \int_1^e (\ln y)(\ln x)\Big|_{x=1}^{x=e^2}\,dy = \int_1^e 2\ln y\,dy = 2(y\ln y - y)\Big|_1^e = 2[(e-e)-(-1)] = 2.$

11. $\int_0^1 \int_0^x (x+2y)\,dy\,dx = \int_0^1 \left(xy+y^2\right)\Big|_{y=0}^{y=x}\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}.$

12. $\int_0^1 \int_0^x xy\,dy\,dx = \int_0^1 \frac{1}{2}xy^2\Big|_{y=0}^{y=x}\,dx = \int_0^1 \frac{1}{2}x^3\,dx = \frac{1}{8}x^4\Big|_0^1 = \frac{1}{8}.$

13. $\int_1^3 \int_0^{x+1} (2x+4y)\,dy\,dx = \int_1^3 \left(2xy+2y^2\right)\Big|_{y=0}^{y=x+1}\,dx = \int_1^3 \left[2x(x+1)+2(x+1)^2\right]dx = \int_1^3 \left(4x^2+6x+2\right)dx$
$= \left(\frac{4}{3}x^3+3x^2+2x\right)\Big|_1^3 = (36+27+6) - \left(\frac{4}{3}+3+2\right) = \frac{188}{3}.$

14. $\int_0^2 \int_{-1}^{1-y} (2-y)\,dx\,dy = \int_0^2 (2x-yx)\Big|_{x=-1}^{x=1-y}\,dy = \int_0^2 \left[2(1-y) - y(1-y) - (-2+y)\right]dy$
$= \int_0^2 \left(4-4y+y^2\right)dy = \left(4y-2y^2+\frac{1}{3}y^3\right)\Big|_0^2 = 8 - 2(4) + \frac{1}{3}(8) = \frac{8}{3}.$

15. $\int_0^4 \int_0^{\sqrt{y}} (x+y)\,dx\,dy = \int_0^4 \left(\frac{1}{2}x^2+xy\right)\Big|_{x=0}^{x=\sqrt{y}}\,dy = \int_0^4 \left(\frac{1}{2}y+y^{3/2}\right)dy = \left(\frac{1}{4}y^2+\frac{2}{5}y^{5/2}\right)\Big|_0^4 = 4+\frac{64}{5} = \frac{84}{5}.$

16. $\int_0^1 \int_{x^3}^{x^2} x^2y^2\,dy\,dx = \int_0^1 \frac{1}{3}x^2y^3\Big|_{y=x^3}^{y=x^2}\,dx = \int_0^1 \frac{1}{3}\left(x^8-x^{11}\right)dx = \frac{1}{3}\left(\frac{1}{9}x^9 - \frac{1}{12}x^{12}\right)\Big|_0^1 = \frac{1}{3}\left(\frac{1}{9}-\frac{1}{12}\right) = \frac{1}{108}.$

17. $\int_0^2 \int_0^{\sqrt{4-y^2}} y\,dx\,dy = \int_0^2 xy\Big|_{x=0}^{x=\sqrt{4-y^2}}\,dy = \int_0^2 y\sqrt{4-y^2}\,dy = -\frac{1}{2}\left(\frac{2}{3}\right)\left(4-y^2\right)^{3/2}\Big|_0^2 = \frac{1}{3}\left(4^{3/2}\right) = \frac{8}{3}.$

18. $\int_0^1 \int_0^x \frac{y}{x^3+2}\,dy\,dx = \int_0^1 \frac{1}{2}\left(\frac{y^2}{x^3+2}\right)\Big|_0^x\,dx = \frac{1}{2}\int_0^1 \frac{x^2}{x^3+2}\,dx = \frac{1}{6}\ln\left(x^3+2\right)\Big|_0^1 = \frac{1}{6}(\ln 3 - \ln 2) = \frac{1}{6}\ln\frac{3}{2}.$

19. $\int_0^1 \int_0^x 2xe^y\,dy\,dx = \int_0^1 2xe^y\Big|_{y=0}^{y=x}\,dx = \int_0^1 (2xe^x - 2x)\,dx = 2(x-1)e^x - x^2\Big|_0^1 = (-1)+2 = 1.$

20. $\int_0^1 \int_y^{e^{2y}} 2x\,dx\,dy = \int_0^1 x^2\Big|_{x=y}^{x=e^{2y}}\,dy = \int_0^1 \left(e^{4y}-y^2\right)dy = \left(\frac{1}{4}e^{4y} - \frac{1}{3}y^3\right)\Big|_0^1 = \frac{1}{4}e^4 - \frac{1}{3} - \frac{1}{4} = \frac{1}{12}\left(3e^4-7\right).$

21. $\int_0^1 \int_x^{\sqrt{x}} ye^x\,dy\,dx = -\int_0^1 \int_{\sqrt{x}}^x ye^x\,dy\,dx = \int_0^1 \left(-\frac{1}{2}y^2e^x\right)\Big|_{y=\sqrt{x}}^{y=x}\,dx = -\frac{1}{2}\int_0^1 \left(x^2e^x - xe^x\right)dx$
$= -\frac{1}{2}\left(x^2e^x\Big|_0^1 - 2\int_0^1 xe^x\,dx - \int_0^1 xe^x\,dx\right) = -\frac{1}{2}\left(x^2e^x\Big|_0^1 - 3\int_0^1 xe^x\,dx\right)$
$= -\frac{1}{2}\left(x^2e^x - 3xe^x + 3e^x\right)\Big|_0^1 = -\frac{1}{2}(e - 3e + 3e - 3) = \frac{1}{2}(3-e).$

22. $\int_0^4 \int_0^{\sqrt{y}} xe^{-y^2}\,dx\,dy = \int_0^4 \frac{1}{2}x^2e^{-y^2}\Big|_{x=0}^{x=\sqrt{y}}\,dy = \frac{1}{2}\int_0^4 ye^{-y^2}\,dy = -\frac{1}{4}e^{-y^2}\Big|_0^4 = -\frac{1}{4}\left(e^{-16}-1\right) = \frac{1}{4}\left(1-e^{-16}\right).$

23. $\int_0^1 \int_{2x}^2 e^{y^2}dy\,dx = \int_0^2 \int_0^{y/2} e^{y^2}\,dx\,dy = \int_0^2 xe^{y^2}\Big|_{x=0}^{x=y/2}\,dy = \int_0^2 \frac{1}{2}ye^{y^2}\,dy = \frac{1}{4}e^{y^2}\Big|_0^2 = \frac{1}{4}\left(e^4-1\right).$

24. $\int_0^{\ln x} \int_1^e y\,dx\,dy = \int_0^{\ln x} yx\Big|_{x=1}^{x=e}dy = \int_0^{\ln x} (e-1)\,y\,dy = (e-1)\,\frac{1}{2}y^2\Big|_0^{\ln x} = \frac{1}{2}(e-1)(\ln x)^2.$

25. $\int_0^2 \int_{y/2}^1 ye^{x^3}\,dx\,dy = \int_0^1 \int_0^{2x} ye^{x^3}\,dy\,dx = \int_0^1 \frac{1}{2}y^2e^{x^3}\Big|_{y=0}^{y=2x}\,dx = \int_0^1 2x^2e^{x^3}\,dx = \frac{2}{3}e^{x^3}\Big|_0^1 = \frac{2}{3}(e-1).$

26. $V = \iint_R f(x,y)\,dA = \int_0^3 \int_0^4 (6-y)\,dy\,dx = \int_0^3 \left[6y - \frac{1}{2}y^2\right]_{y=0}^{y=4} dx = \int_0^3 16\,dx = 48$

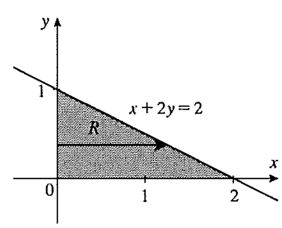
27. $V = \int_0^4 \int_0^3 \left(4 - x + \frac{1}{2}y\right) dx\,dy = \int_0^4 \left(4x - \frac{1}{2}x^2 + \frac{1}{2}xy\right)\Big|_{x=0}^{x=3} dy = \int_0^4 \left(\frac{15}{2} + \frac{3}{2}y\right) dy$
$= \left(\frac{15}{2}y + \frac{3}{4}y^2\right)\Big|_0^4 = 42.$

28. $V = \int_0^2 \int_x^{4-x} 5\,dy\,dx = \int_0^2 5y\big|_{y=x}^{y=4-x} dx = 5\int_0^2 (4-2x)\,dx = 5\left(4x - x^2\right)\big|_0^2 = 20.$

29. $V = \int_0^2 \int_0^{3-(3/2)z} (6 - 2y - 3z)\,dy\,dz = \int_0^2 \left(6y - y^2 - 3yz\right)\big|_{y=0}^{y=3-(3/2)z} dz$
$= \int_0^2 \left[6\left(3 - \frac{3}{2}z\right) - \left(3 - \frac{3}{2}z\right)^2 - 3\left(3 - \frac{3}{2}z\right)z\right] dz = \left[-2\left(3 - \frac{3}{2}z\right)^2 - \frac{2}{9}\left(3 - \frac{3}{2}z\right)^3 - \frac{9}{2}z^2 + \frac{3}{2}z^3\right]_0^2$
$= (-18 + 12) - (-18 + 6) = 6.$

30. $V = \iint_R f(x,y)\,dA = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \left(4 - x^2 - y^2\right) dy\,dx = 4\int_0^2 \left[4y - x^2y - \frac{1}{3}y^3\right]_{y=0}^{y=\sqrt{4-x^2}} dx$
$= 4\int_0^2 \left[\left(4 - x^2\right)\sqrt{4 - x^2} - \frac{1}{3}\left(4 - x^2\right)^{3/2}\right] dx = \frac{8}{3}\int_0^2 \left(2^2 - x^2\right)^{3/2} dx.$
Using a calculator to evaluate the last integral, we find that $V = 8\pi \approx 25.1327$.

31. $V = \iint_R f(x,y)\,dA = \int_0^1 \int_0^{2-2y} \left(4 - x^2 - y^2\right) dx\,dy$
$= \int_0^1 \left[\left(4 - y^2\right)x - \frac{1}{3}x^3\right]_{x=0}^{x=2-2y} dy$
$= \int_0^1 \left[\left(4 - y^2\right)(2 - 2y) - \frac{1}{3}(2 - 2y)^3\right] dy$
$= \int_0^1 \left(\frac{14}{3}y^3 - 10y^2 + \frac{16}{3}\right) dy = \left(\frac{7}{6}y^4 - \frac{10}{3}y^3 + \frac{16}{3}y\right)\Big|_0^1 = \frac{19}{6}$



32. $V = \iint_R f(x,y)\,dA = \int_0^4 \int_0^2 \left(4 - x^2\right) dx\,dy = \int_0^4 \left[4x - \frac{1}{3}x^3\right]_0^2 dy = \int_0^4 \frac{16}{3}\,dy = \frac{64}{3}$

33. $V = \iint_R f(x,y)\,dA = \int_0^2 \int_0^2 2e^{-x}e^{-y}\,dx\,dy = \int_0^2 \left[-2e^{-x}e^{-y}\right]_{x=0}^{x=2} dy = \int_0^2 \left(-2e^{-2}e^{-y} + 2e^{-y}\right) dy$
$= \left(2e^{-2}e^{-y} - 2e^{-y}\right)\big|_0^2 = \dfrac{2\left(e^2 - 1\right)^2}{e^4}$

34. $V = \int_0^1 \int_0^2 (4 - 2x - y)\,dy\,dx = \int_0^1 \left(4y - 2xy - \frac{1}{2}y^2\right)\Big|_{y=0}^{y=2} dx = \int_0^1 (8 - 4x - 2)\,dx = \int_0^1 (6 - 4x)\,dx$
$= \left(6x - 2x^2\right)\big|_0^1 = 6 - 2 = 4.$

35. $V = \int_0^2 \int_0^{2x} (2x + y)\,dy\,dx = \int_0^2 \left(2xy + \frac{1}{2}y^2\right)\Big|_0^{2x} dx = \int_0^2 \left(4x^2 + 2x^2\right) dx = \int_0^2 6x^2\,dx = 2x^3\big|_0^2 = 16.$

36. $V = \int_0^1 \int_0^2 \left(x^2 + y^2\right) dy\,dx = \int_0^1 \left(x^2y + \frac{1}{3}y^3\right)\Big|_{y=0}^{y=2} dx = \int_0^1 \left(2x^2 + \frac{8}{3}\right) dx = \left(\frac{2}{3}x^3 + \frac{8}{3}x\right)\Big|_0^1$
$= \frac{2}{3} + \frac{8}{3} = \frac{10}{3}.$

37. $V = \int_0^1 \int_0^{-x+1} e^{x+2y}\,dy\,dx = \int_0^1 \frac{1}{2}e^{x+2y}\Big|_{y=0}^{y=-x+1}\,dx = \frac{1}{2}\int_0^1 \left(e^{-x+2} - e^x\right)dx = \frac{1}{2}\left(-e^{-x+2} - e^x\right)\Big|_0^1$
$= \frac{1}{2}\left(-e - e + e^2 + 1\right) = \frac{1}{2}\left(e^2 - 2e + 1\right) = \frac{1}{2}(e-1)^2.$

38. $V = \int_0^2 \int_x^2 2xe^y\,dy\,dx = \int_0^2 2xe^y\big|_{y=x}^{y=2}\,dx = \int_0^2 \left(2xe^2 - 2xe^x\right)dx = \left[e^2x^2 - 2(x-1)e^x\right]_0^2$ (by parts)
$= 4e^2 - 2e^2 - 2 = 2\left(e^2 - 1\right).$

39. $V = \displaystyle\int_0^4 \int_0^{\sqrt{x}} \frac{2y}{1+x^2}\,dy\,dx = \int_0^4 \frac{y^2}{1+x^2}\Big|_0^{\sqrt{x}}\,dx = \int_0^4 \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln\left(1+x^2\right)\Big|_0^4 = \frac{1}{2}(\ln 17 - \ln 1) = \frac{1}{2}\ln 17.$

40. $V = \int_0^1 \int_{x^2}^x 2x^2y\,dy\,dx = \int_0^1 x^2y^2\big|_{y=x^2}^{y=x}\,dx = \int_0^1 \left(x^4 - x^6\right)dx = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}.$

41. $V = \int_0^4 \int_0^{\sqrt{16-x^2}} x\,dy\,dx = \int_0^4 xy\big|_{y=0}^{y=\sqrt{16-x^2}}\,dx = \int_0^4 x\left(16-x^2\right)^{1/2}dx = \left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(16-x^2\right)^{3/2}\Big|_0^4$
$= \frac{1}{3}(16)^{3/2} = \frac{64}{3}.$

42. $A = \frac{1}{6}\int_0^3 \int_0^2 6x^2y^3\,dx\,dy = \int_0^3 \frac{1}{3}x^3y^3\Big|_0^2\,dy = \frac{8}{3}\int_0^3 y^3\,dy = \frac{2}{3}y^4\Big|_0^3 = 54.$

43. $A = \frac{1}{1/2}\int_0^1 \int_0^x (x+2y)\,dy\,dx = 2\int_0^1 \left(xy+y^2\right)\big|_0^x\,dx = 2\int_0^1 \left(x^2+x^2\right)dx = 4\int_0^1 x^2\,dx = \frac{4}{3}x^3\Big|_0^1 = \frac{4}{3}.$

44. The area of R is $\frac{1}{2}(2)(1) = 1$, so the average value of f is
$\int_0^1 \int_y^{2-y} xy\,dx\,dy = \int_0^1 \frac{1}{2}x^2y\Big|_{x=y}^{x=2-y}\,dy = \int_0^1 \left[\frac{1}{2}(2-y)^2y - \frac{1}{2}y^3\right]dy = \int_0^1 \left(2y - 2y^2\right)dy = \left(y^2 - \frac{2}{3}y^3\right)\Big|_0^1 = \frac{1}{3}.$

45. The area of R is $\frac{1}{2}$, so the average value of f is
$\frac{1}{1/2}\int_0^1 \int_0^x e^{-x^2}\,dy\,dx = 2\int_0^1 e^{-x^2}y\Big|_{y=0}^{y=x}\,dx = 2\int_0^1 xe^{-x^2}\,dx = -e^{-x^2}\Big|_0^1 = -e^{-1} + 1 = 1 - \frac{1}{e}.$

46. The area of R is $\frac{1}{2}$, so the average value of f is
$2\int_0^1 \int_0^x xe^y\,dy\,dx = 2\int_0^1 xe^y\big|_{y=0}^{y=x}\,dx = 2\int_0^1 \left(xe^x - x\right)dx = 2\left(xe^x - e^x - \frac{1}{2}x^2\right)\Big|_0^1 = 2\left(e - e - \frac{1}{2} + 1\right) = 1.$

47. By elementary geometry, the area of the region is $4 + \frac{1}{2}(2)(4) = 8$. Therefore, the required average value is
$A = \frac{1}{8}\int_1^3 \int_0^{2x} \ln x\,dy\,dx = \frac{1}{8}\int_1^3 (\ln x)\,y\big|_{y=0}^{y=2x}\,dx = \frac{1}{4}\int_1^3 x\ln x\,dx = \frac{1}{4}\left(\frac{1}{4}x^2\right)(2\ln x - 1)\Big|_1^3$ (by parts)
$= \frac{9}{16}(2\ln 3 - 1) - \frac{1}{16}(-1) = \frac{1}{8}(9\ln 3 - 4).$

48. The population is
$$2\int_0^5 \int_{-2}^0 \frac{10{,}000e^y}{1+0.5x}\,dy\,dx = 20{,}000\int_0^5 \frac{e^y}{1+0.5x}\Big|_{y=-2}^{y=0}\,dx = 20{,}000\left(1-e^{-2}\right)\int_0^5 \frac{1}{1+0.5x}\,dx$$
$$= 20{,}000\left(1-e^{-2}\right)\cdot 2\ln(1+0.5x)\big|_0^5 = 40{,}000\left(1-e^{-2}\right)\ln 3.5 \approx 43{,}329.$$

49. The average population density inside R is $\frac{43{,}329}{20} \approx 2166$ people per square mile.

50. By symmetry, it suffices to compute the population in the first quadrant. In the first quadrant, $f(x,y) = \dfrac{50{,}000xy}{(x^2+20)(y^2+36)}$. Therefore, the population in R is given by

$$\int_R\int f(x,y)\,dA = 4\int_0^{15}\left[\int_0^{20}\frac{50{,}000xy}{(x^2+20)(y^2+36)}\,dy\right]dx = 4\int_0^{15}\left[\frac{50{,}000x\left(\frac{1}{2}\right)\ln(y^2+36)}{x^2+20}\right]_0^{20}dx$$

$$= 100{,}000(\ln 436 - \ln 36)\int_0^{15}\frac{x}{x^2+20}\,dx = 100{,}000(\ln 436 - \ln 36)\left(\tfrac{1}{2}\right)\ln(x^2+20)\Big|_0^{15}$$

$$= 50{,}000(\ln 436 - \ln 36)(\ln 245 - \ln 20) \approx 312{,}455,$$ or approximately 312,455 people.

51. The average weekly profit is

$$P = \frac{1}{(20)(20)}\int_{100}^{120}\int_{180}^{200}\left(-0.2x^2 - 0.25y^2 - 0.2xy + 100x + 90y - 4000\right)dx\,dy$$

$$= \frac{1}{400}\int_{100}^{120}\left(-\tfrac{1}{15}x^3 - 0.25y^2x - 0.1x^2y + 50x^2 + 90xy - 4000x\right)\Big|_{x=180}^{x=200}dy$$

$$= \frac{1}{400}\int_{100}^{120}\left(-144{,}533.33 - 5y^2 - 760y + 380{,}000 + 1800y - 80{,}000\right)dy$$

$$= \frac{1}{400}\int_{100}^{120}\left(155{,}466.67 - 5y^2 + 1040y\right)dy = \frac{1}{400}\left(155{,}466.67y - \tfrac{5}{3}y^3 + 520y^2\right)\Big|_{100}^{120}$$

$$= \frac{1}{400}(3{,}109{,}333.40 - 1{,}213{,}333.30 + 2{,}288{,}000) \approx 10{,}460,$$ or \$10,460.

52. The average price is

$$P = \tfrac{1}{2}\int_0^1\int_0^2\left[200 - 10\left(x-\tfrac{1}{2}\right)^2 - 15(y-1)^2\right]dy\,dx = \tfrac{1}{2}\int_0^1\left[200y - 10\left(x-\tfrac{1}{2}\right)^2y - 5(y-1)^3\right]_0^2dx$$

$$= \tfrac{1}{2}\int_0^1\left[400 - 20\left(x-\tfrac{1}{2}\right)^2 - 5 - 5\right]dx = \tfrac{1}{2}\int_0^1\left[390 - 20\left(x-\tfrac{1}{2}\right)^2\right]dx = \tfrac{1}{2}\left[390x - \tfrac{20}{3}\left(x-\tfrac{1}{2}\right)^3\right]_0^1$$

$$= \tfrac{1}{2}\left[390 - \tfrac{20}{3}\left(\tfrac{1}{8}\right) - \tfrac{20}{3}\left(\tfrac{1}{8}\right)\right] \approx 194.17,$$ or approximately \$194 per square foot.

53. True. This result follows from the definition.

54. False. Let $f(x,y) = \dfrac{x}{y-2}$, $a = 0$, $b = 3$, $c = 0$, and $d = 1$. Then $\int_{R_1}\int f(x,y)\,dA$ is defined on $R_1 = \{(x,y) \mid 0 \le x \le 3, 0 \le y \le 1\}$, but $\int_{R_2}\int f(x,y)\,dA$ is not defined on $R_2 = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 3\}$, because f is discontinuous on R_2 where $y = 2$.

55. True. $\int_R\int g(x,y)\,dA$ gives the volume of the solid bounded above by the surface $z = g(x,y)$ and $\int_R\int f(x,y)\,dA$ gives the volume of the solid bounded above by the surface $z = f(x,y)$. Therefore, $\int_R\int g(x,y)\,dA - \int_R\int f(x,y)\,dA = \int_R\int[g(x,y) - f(x,y)]\,dA$ gives the volume of the solid bounded above by $z = g(x,y)$ and below by $z = f(x,y)$.

56. True. The average value is $V_A = \dfrac{\int_R\int f(x,y)\,dA}{\int_R\int dA}$, and so $V_A\int_R\int dA = \int_R\int f(x,y)\,dA$. The quantity on the left-hand side is the volume of such a cylinder.

CHAPTER 8 Concept Review Questions page 635

1. xy, ordered pair, real number, $f(x,y)$

2. independent, dependent, value

3. $z = f(x, y)$, f, surface

4. $f(x, y) = k$, level curve, level curves, k

5. constant, x

6. slope, $(a, b, f(a, b))$, x, b

7. $\leq$, (a, b), $\leq$, domain

8. domain, $f_x(a, b) = 0$ and $f_y(a, b) = 0$, exist, candidate

9. scatter, minimizing, least-squares, normal

10. $g(x, y) = 0$, $f(x, y) + \lambda g(x, y)$, $F_x = 0$, $F_y = 0$, $F_\lambda = 0$, extrema

11. volume, solid

12. iterated, $\int_3^5 \int_0^1 (2x + y^2)\, dx\, dy$

CHAPTER 8 Review Exercises page 635

1. $f(x, y) = \dfrac{xy}{x^2 + y^2}$, so $f(0, 1) = 0$, $f(1, 0) = 0$, $f(1, 1) = \dfrac{1}{1+1} = \dfrac{1}{2}$, and $f(0, 0)$ does not exist because the point $(0, 0)$ does not lie in the domain of f.

2. $f(x, y) = \dfrac{xe^y}{1 + \ln xy}$, so $f(1, 1) = \dfrac{e}{1 + \ln 1} = e$, $f(1, 2) = \dfrac{e^2}{1 + \ln 2}$, $f(2, 1) = \dfrac{2e}{1 + \ln 2}$, and $f(1, 0)$ does not exist because the point $(0, 0)$ does not lie in the domain of f.

3. $h(x, y, z) = xye^z + \dfrac{x}{y}$, so $h(1, 1, 0) = 1 + 1 = 2$, $h(-1, 1, 1) = -e - 1 = -(e + 1)$, and $h(1, -1, 1) = -e - 1 = -(e + 1)$.

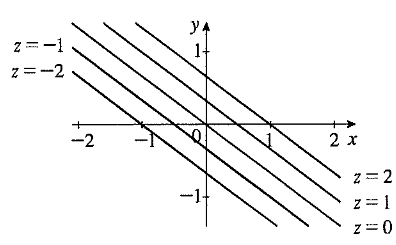
4. $f(u, v) = \dfrac{\sqrt{u}}{u - v}$. The domain of f is the set of all ordered pairs (u, v) of real numbers such that $u \geq 0$ and $u \neq v$.

5. $f(x, y) = \dfrac{x - y}{x + y}$, so $D = \{(x, y) \mid y \neq -x\}$.

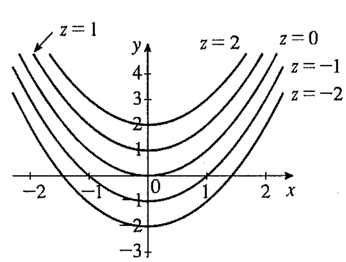
6. $f(x, y) = x\sqrt{y} + y\sqrt{1 - x}$, so $D = \{(x, y) \mid x \leq 1, y \geq 0\}$.

7. $f(x, y, z) = \dfrac{xy\sqrt{z}}{(1 - x)(1 - y)(1 - z)}$. The domain of f is the set of all ordered triples (x, y, z) of real numbers such that $z \geq 0$, $x \neq 1$, $y \neq 1$, and $z \neq 1$.

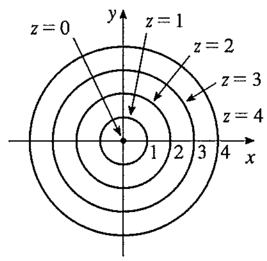
8. $2x + 3y = z$



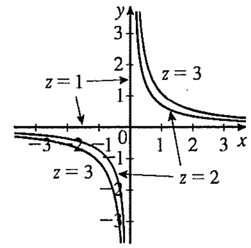
9. $z = y - x^2$



10. $z = \sqrt{x^2 + y^2}$



11. $z = e^{xy}$



12. $f(x, y) = x^2y^3 + 3xy^2 + \dfrac{x}{y}$, so $f_x = 2xy^3 + 3y^2 + \dfrac{1}{y}$ and $f_y = 3x^2y^2 + 6xy - \dfrac{x}{y^2}$.

13. $f(x, y) = x\sqrt{y} + y\sqrt{x}$, so $f_x = \sqrt{y} + \dfrac{y}{2\sqrt{x}}$ and $f_y = \dfrac{x}{2\sqrt{y}} + \sqrt{x}$.

14. $f(u, v) = \sqrt{uv^2 - 2u}$, so $f_u = \frac{1}{2}\left(uv^2 - 2u\right)^{-1/2}\left(v^2 - 2\right) = \dfrac{v^2 - 2}{2\sqrt{uv^2 - 2u}}$ and
$f_v = \frac{1}{2}\left(uv^2 - 2u\right)^{-1/2}(2uv) = \dfrac{uv}{\sqrt{uv^2 - 2u}}$.

15. $f(x, y) = \dfrac{x - y}{y + 2x}$, so $f_x = \dfrac{(y + 2x) - (x - y)(2)}{(y + 2x)^2} = \dfrac{3y}{(y + 2x)^2}$ and
$f_y = \dfrac{(y + 2x)(-1) - (x - y)}{(y + 2x)^2} = \dfrac{-3x}{(y + 2x)^2}$.

16. $g(x, y) = \dfrac{xy}{x^2 + y^2}$, so $g_x = \dfrac{\left(x^2 + y^2\right)y - xy(2x)}{\left(x^2 + y^2\right)^2} = \dfrac{y^3 - x^2y}{\left(x^2 + y^2\right)^2} = \dfrac{y\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2} = \dfrac{y(y - x)(y + x)}{\left(x^2 + y^2\right)^2}$ and
$g_y = \dfrac{\left(x^2 + y^2\right)x - xy(2y)}{\left(x^2 + y^2\right)^2} = \dfrac{x\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} = \dfrac{x(x - y)(x + y)}{\left(x^2 + y^2\right)^2}$.

17. $h(x, y) = \left(2xy + 3y^2\right)^5$, so $h_x = 10y\left(2xy + 3y^2\right)^4$ and $h_y = 10(x + 3y)\left(2xy + 3y^2\right)^4$.

18. $f(x, y) = (xe^y + 1)^{1/2}$, so $f_x = \frac{1}{2}(xe^y + 1)^{-1/2}e^y = \dfrac{e^y}{2(xe^y + 1)^{1/2}}$ and
$f_y = \frac{1}{2}(xe^y + 1)^{-1/2}xe^y = \dfrac{xe^y}{2(xe^y + 1)^{1/2}}$.

19. $f(x, y) = \left(x^2 + y^2\right)e^{x^2+y^2}$, so $f_x = 2xe^{x^2+y^2} + \left(x^2 + y^2\right)(2x)e^{x^2+y^2} = 2x\left(x^2 + y^2 + 1\right)e^{x^2+y^2}$ and
$f_y = 2ye^{x^2+y^2} + \left(x^2 + y^2\right)(2y)e^{x^2+y^2} = 2y\left(x^2 + y^2 + 1\right)e^{x^2+y^2}$.

20. $f(x, y) = \ln\left(1 + 2x^2 + 4y^4\right)$, so $f_x = \dfrac{4x}{1 + 2x^2 + 4y^4}$ and $f_y = \dfrac{16y^3}{1 + 2x^2 + 4y^4}$.

21. $f(x, y) = \ln\left(1 + \dfrac{x^2}{y^2}\right)$, so $f_x = \dfrac{2x/y^2}{1 + \left(x^2/y^2\right)} = \dfrac{2x}{x^2 + y^2}$ and $f_y = \dfrac{-2x^2/y^3}{1 + \left(x^2/y^2\right)} = -\dfrac{2x^2}{y\left(x^2 + y^2\right)}$.

22. $f(x,y)=x^3-2x^2y+y^2+x-2y$, so $f_x=3x^2-4xy+1$ and $f_y=-2x^2+2y-2$. Therefore, $f_{xx}=6x-4y$, $f_{xy}=f_{yx}=-4x$, and $f_{yy}=2$.

23. $f(x,y)=x^4+2x^2y^2-y^4$, so $f_x=4x^3+4xy^2$ and $f_y=4x^2y-4y^3$. Therefore, $f_{xx}=12x^2+4y^2$, $f_{xy}=8xy=f_{yx}$, and $f_{yy}=4x^2-12y^2$.

24. $f(x,y)=\left(2x^2+3y^2\right)^3$, so $f_x=3\left(2x^2+3y^2\right)^2(4x)=12x\left(2x^2+3y^2\right)^2$ and $f_y=3\left(2x^2+3y^2\right)^2(6y)=18y\left(2x^2+3y^2\right)^2$. Therefore,

$$\begin{aligned}f_{xx}&=12\left(2x^2+3y^2\right)^2+12x(2)\left(2x^2+3y^2\right)(4x)=12\left(2x^2+3y^2\right)^2\left[\left(2x^2+3y^2\right)+8x^2\right]\\&=12\left(2x^2+3y^2\right)\left(10x^2+3y^2\right),\end{aligned}$$

$$f_{xy}=12x(2)\left(2x^2+3y^2\right)(6y)=144xy\left(2x^2+3y^2\right)=f_{yx},\text{ and}$$

$$\begin{aligned}f_{yy}&=18\left(2x^2+3y^2\right)^2+18y(2)\left(2x^2+3y^2\right)(6y)=18\left(2x^2+3y^2\right)\left[\left(2x^2+3y^2\right)+12y^2\right]\\&=18\left(2x^2+3y^2\right)\left(2x^2+15y^2\right).\end{aligned}$$

25. $g(x,y)=\dfrac{x}{x+y^2}$, so $g_x=\dfrac{\left(x+y^2\right)-x}{\left(x+y^2\right)^2}=\dfrac{y^2}{\left(x+y^2\right)^2}$ and

$$g_y=\frac{-2xy}{\left(x+y^2\right)^2}.\text{ Therefore, } g_{xx}=-2y^2\left(x+y^2\right)^{-3}=-\frac{2y^2}{\left(x+y^2\right)^3},$$

$$g_{xy}=\frac{\left(x+y^2\right)2y-y^2(2)\left(x+y^2\right)2y}{\left(x+y^2\right)^4}=\frac{2\left(x+y^2\right)\left(xy+y^3-2y^3\right)}{\left(x+y^2\right)^4}=\frac{2y\left(x-y^2\right)}{\left(x+y^2\right)^3}=g_{yx},\text{ and}$$

$$g_{yy}=\frac{\left(x+y^2\right)^2(-2x)+2xy(2)\left(x+y^2\right)2y}{\left(x+y^2\right)^4}=\frac{2x\left(x^2+y^2\right)\left(-x-y^2+4y^2\right)}{\left(x+y^2\right)^4}=\frac{2x\left(3y^2-x\right)}{\left(x+y^2\right)^3}$$

26. $g(x,y)=e^{x^2+y^2}$, so $g_x=2xe^{x^2+y^2}$ and $g_y=2ye^{x^2+y^2}$. Therefore, $g_{xx}=2e^{x^2+y^2}+(2x)^2e^{x^2+y^2}=2\left(1+2x^2\right)e^{x^2+y^2}$, $g_{xy}=4xye^{x^2+y^2}=g_{yx}$, and $g_{yy}=2e^{x^2+y^2}+(2y)^2e^{x^2+y^2}=2\left(1+2y^2\right)e^{x^2+y^2}$.

27. $h(s,t)=\ln\left(\dfrac{s}{t}\right)$. Write $h(s,t)=\ln s-\ln t$. Then $h_s=\dfrac{1}{s}$ and $h_t=-\dfrac{1}{t}$, so $h_{ss}=-\dfrac{1}{s^2}$, $h_{st}=h_{ts}=0$, and $h_{tt}=\dfrac{1}{t^2}$.

28. $f(x,y,z)=x^3y^2z+xy^2z+3xy-4z$, so $f_x(1,1,0)=\left(3x^2yz+y^2z+3y\right)\big|_{(1,1,0)}=3$; $f_y(1,1,0)=\left(2x^3yz+2xyz+3x\right)\big|_{(1,1,0)}=3$, and $f_z(1,1,0)=\left(x^3y^2+xy^2-4\right)\big|_{(1,1,0)}=-2$.

29. $f(x,y)=2x^2+y^2-8x-6y+4$. To find the critical points of f, we solve the system $\begin{cases}f_x=4x-8=0\\f_y=2y-6=0\end{cases}$ obtaining $x=2$ and $y=3$. Therefore, the sole critical point of f is $(2,3)$. Next, $f_{xx}=4$, $f_{xy}=0$, and $f_{yy}=2$, so $D(2,3)=f_{xx}(2,3)f_{yy}(2,3)-f_{xy}(2,3)^2=8>0$. Because $f_{xx}(2,3)>0$, we see that $f(2,3)=-13$ is a relative minimum value.

30. $f(x,y) = x^2 + 3xy + y^2 - 10x - 20y + 12$. We solve the system $\begin{cases} f_x = 2x + 3y - 10 = 0 \\ f_y = 3x + 2y - 20 = 0 \end{cases}$ obtaining $x = 8$ and $y = -2$, so $(8, -2)$ is the only critical point of f. Next, we compute $f_{xx} = 2$, $f_{xy} = 3$, and $f_{yy} = 2$, so $D(8,-2) = f_{xx}(8,-2) f_{yy}(8,-2) - f_{xy}^2(8,-2) = (2)(2) - 3^2 = -5 < 0$. Because $D < 0$, we see that $(8, -2)$ gives rise to a saddle point of f. Because $f(8,-2) = -8$, the saddle point is $(8, -2, -8)$.

31. $f(x,y) = x^3 - 3xy + y^2$. We solve the system of equations $\begin{cases} f_x = 3x^2 - 3y = 0 \\ f_y = -3x + 2y = 0 \end{cases}$ obtaining $x^2 - y = 0$, and so $y = x^2$. Then $-3x + 2x^2 = 0$, $x(2x-3) = 0$, and so $x = 0$ or $x = \frac{3}{2}$. The corresponding values of y are $y = 0$ and $y = \frac{9}{4}$, so the critical points are $(0,0)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$. Next, $f_{xx} = 6x$, $f_{xy} = -3$, and $f_{yy} = 2$, and so $D(x,y) = 12x - 9 = 3(4x-3)$. Therefore, $D(0,0) = -9$, and so $(0,0)$ is a saddle point and $f(0,0) = 0$. $D\left(\frac{3}{2}, \frac{9}{4}\right) = 3(6-3) = 9 > 0$ and $f_{xx}\left(\frac{3}{2}, \frac{9}{4}\right) > 0$, and so $f\left(\frac{3}{2}, \frac{9}{4}\right) = \frac{27}{8} - \frac{81}{8} + \frac{81}{16} = -\frac{27}{16}$ is a relative minimum value.

32. $f(x,y) = x^3 + y^2 - 4xy + 17x - 10y + 8$. To find the critical points of f, we solve the system $\begin{cases} f_x = 3x^2 - 4y + 17 = 0 \\ f_y = 2y - 4x - 10 = 0 \end{cases}$ From the second equation, we have $y = 2x + 5$ which, when substituted into the first equation, gives $3x^2 - 8x - 20 + 17 = 0$, so $3x^2 - 8x - 3 = (3x+1)(x-3) = 0$. The solutions are $x = -\frac{1}{3}$ and $x = 3$, so the critical points of f are $\left(-\frac{1}{3}, \frac{13}{3}\right)$ and $(3, 11)$. Next, we compute $f_{xx} = 6x$, $f_{xy} = -4$, and $f_{yy} = 2$, and so $D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 12x - 16$. Because $D\left(-\frac{1}{3}, \frac{13}{3}\right) = -20 < 0$, we see that $\left(-\frac{1}{3}, \frac{13}{3}\right)$ gives a saddle point and $f\left(-\frac{1}{3}, \frac{13}{3}\right) = -\frac{445}{27}$; and because $D(3,11) = 20 > 0$ and $f_{xx}(3,11) = 18 > 0$, we see that $(3, 11)$ gives a relative minimum value of f, namely $f(3,11) = -35$.

33. $f(x,y) = f(x,y) = e^{2x^2+y^2}$. To find the critical points of f, we solve the system $\begin{cases} f_x = 4xe^{2x^2+y^2} = 0 \\ f_y = 2ye^{2x^2+y^2} = 0 \end{cases}$ giving $(0,0)$ as the only critical point of f. Next, $f_{xx} = 4\left(e^{2x^2+y^2} + 4x^2e^{2x^2+y^2}\right) = 4\left(1+4x^2\right)e^{2x^2+y^2}$, $f_{xy} = 8xye^{2x^2+y^2} = f_{yx}$, and $f_{yy} = 2\left(1+2y^2\right)e^{2x^2+y^2}$, so $D = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0) = (4)(2) - 0 = 8 > 0$. Because $f_{xx}(0,0) > 0$, we see that $(0,0)$ gives a relative minimum of f. The minimum value of f is $f(0,0) = e^0 = 1$.