5.6 Exponential Functions as Mathematical Models

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1. $Q(t) = Q_0 e^{kt}$ where $k > 0$ represents exponential growth and $k < 0$ represents exponential decay. The larger the magnitude of k, the more quickly the former grows and the more quickly the latter decays.

2. The half-life of a radioactive substance is the time it takes for the substance to decay to half its original amount.

3. $Q(t) = \dfrac{A}{1 + Be^{-kt}}$, where A, B, and k are positive constants. Q increases rapidly for small values of t but the rate of increase slows down as Q (always increasing) approaches the number A.

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1. **a.** The growth constant is $k = 0.02$.

b. Initially, there are 300 units present.

c.

t	0	10	20	100	1000
Q	300	366	448	2217	1.46×10^{11}

2. **a.** $k = -0.06$.

b. $Q_0 = 2000$.

c.

t	0	5	10	20	100
Q	2000	1482	1098	602	5

3. **a.** $Q(t) = Q_0 e^{kt}$. Here $Q_0 = 100$ and so $Q(t) = 100e^{kt}$. Because the number of cells doubles in 20 minutes, we have $Q(20) = 100e^{20k} = 200$, $e^{20k} = 2$, $20k = \ln 2$, and so $k = \frac{1}{20}\ln 2 \approx 0.03466$. Thus, $Q(t) = 100e^{0.03466t}$.

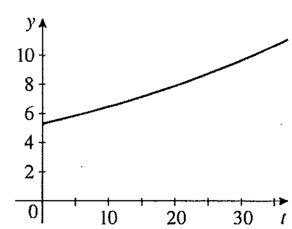
b. We solve the equation $100e^{0.03466t} = 1{,}000{,}000$, obtaining $e^{0.03466t} = 10{,}000$, $0.03466t = \ln 10{,}000$, and so $t = \dfrac{\ln 10{,}000}{0.03466} \approx 266$, or 266 minutes.

c. $Q(t) = 1000e^{0.03466t}$.

4. $Q(t) = 5.3e^{kt}$. Because the population grows at the rate of 2% per year, we have $N(t) = 5.3e^{0.02t}$. Note that $t = 0$ corresponds to the beginning of 1990.

a.

t	$Q(t)$
1990	5.3
1995	5.86
2000	6.47
2005	7.15
2010	7.91
2015	8.74
2020	9.66
2025	10.67



b. $Q'(t) = 5.3(0.02)e^{0.02t} = 0.106e^{0.02t}$, so the rate of growth in the year 2010 is $Q'(20) = 0.106e^{0.02(20)} \approx 0.1581$, or approximately 0.16 billion people per year.

5. **a.** We solve the equation $5.3e^{0.02t} = 3(5.3)$, obtaining $e^{0.02t} = 3$, $0.02t = \ln 3$, and so $t = \dfrac{\ln 3}{0.02} \approx 54.93$. Thus, the world population will triple in approximately 54.93 years.

b. If the growth rate is 1.8%, then proceeding as before, we find $N(t) = 5.3e^{0.018t}$. If $t = 54.93$, the population would be $N(54.93) = 5.3e^{0.018(54.93)} \approx 14.25$, or approximately 14.25 billion.

6. The resale value of the machinery at any time t is given by $V(t) = 500{,}000e^{-kt}$, where $t = 0$ represents three years ago. We have $V(3) = 320{,}000 = 500{,}000e^{-3k}$, which gives $e^{-3k} = \dfrac{320{,}000}{500{,}000} = 0.64$. Therefore, $-3k \ln e = \ln 0.64$ and $k = \dfrac{\ln 0.64}{-3} \approx 0.149$. Four years from now, the resale value of the machinery will be $V(7) = 500{,}000e^{-(0.149)(7)} \approx 176{,}198$, or approximately \$176,198.

7. $P(h) = p_0e^{-kh}$, so $P(0) = p_0 = 15$. Thus, $P(4000) = 15e^{-4000k} = 12.5$, $e^{-4000k} = \frac{12.5}{15}$, $-4000k = \ln\left(\frac{12.5}{15}\right)$, and so $k = 0.00004558$. Therefore, $P(12{,}000) = 15e^{-0.00004558(12{,}000)} = 8.68$, or 8.7 lb/in^2. The rate of change of atmospheric pressure with respect to altitude is given by $P'(h) = \dfrac{d}{dh}\left(15e^{-0.00004558h}\right) = -0.0006837e^{-0.00004558h}$. Thus, the rate of change of atmospheric pressure with respect to altitude when the altitude is 12,000 feet is $P'(12{,}000) = -0.0006837e^{-0.00004558(12{,}000)} \approx -0.00039566$. That is, it is declining at the rate of approximately 0.0004 lb/in^2/ft.

8. We are given that $Q(280) = 20$. Using this condition, we have $Q(280) = Q_0 \cdot 2^{-280/140} = 20$. Thus, $Q_0 \cdot 2^{-2} = 20$, so $\frac{1}{4}Q_0 = 20$ and $Q_0 = 80$. Thus, the initial amount was 80 mg.

9. Suppose the amount of P-32 at time t is given by $Q(t) = Q_0e^{-kt}$, where Q_0 is the amount present initially and k is the decay constant. Because this element has a half-life of 14.2 days, we have $\frac{1}{2}Q_0 = Q_0e^{-14.2k}$, so $e^{-14.2k} = \frac{1}{2}$, $-14.2k = \ln\frac{1}{2}$, and $k = -\frac{\ln(1/2)}{14.2} \approx 0.0488$. Therefore, the amount of P-32 present at any time t is given by $Q(t) = 100e^{-0.0488t}$. In particular, the amount left after 7.1 days is given by $Q(7.1) = 100e^{-0.0488(7.1)} = 100e^{-0.34648} \approx 70.717$, or 70.717 grams. The rate at which the element decays is $Q'(t) = \dfrac{d}{dt}\left(100e^{-0.0488t}\right) = 100(-0.0488)e^{-0.0488t} = -4.88e^{-0.0488t}$. Therefore, $Q'(7.1) = -4.88e^{-0.0488(7.1)} \approx -3.451$; that is, it is decreasing at the rate of 3.451 g/day.

10. Suppose the amount of Sr-90 present at time t is given by $Q(t) = Q_0e^{-kt}$, where Q_0 is the amount present initially and k is the decay constant. Because this element has a half-life of 27 years, we find $\frac{1}{2}Q_0 = Q_0e^{-27k}$, $e^{-27k} = \frac{1}{2}$, $-27k = \ln\frac{1}{2}$, and so $k = -\frac{1}{27}\ln\frac{1}{2}$. Therefore, the amount of Sr-90 present at time t is given by $Q(t) = Q_0e^{(1/27)\ln(1/2)\cdot t} = Q_0e^{(t/27)\ln(1/2)} = Q_0\left(\frac{1}{2}\right)^{t/27}$. To find t when $Q(t) = \frac{1}{4}Q_0$, we calculate $\frac{1}{4}Q_0 = Q_0\left(\frac{1}{2}\right)^{t/27}$, $\left(\frac{1}{2}\right)^{t/27} = \frac{1}{4}$, $\frac{1}{27}t\ln\frac{1}{2} = \ln\frac{1}{4}$, $t = 27\dfrac{\ln\frac{1}{4}}{\ln\frac{1}{2}} = 27\left(\dfrac{-\ln 4}{-\ln 2}\right) \approx 54$, or approximately 54 years.

11. We solve the equation $0.2Q_0 = Q_0e^{-0.00012t}$, obtaining $\ln 0.2 = -0.00012t$ and $t = \dfrac{\ln 0.2}{-0.00012} \approx 13{,}412$, or approximately 13,412 years.

12. We solve the equation $0.18Q_0 = Q_0e^{-0.00012t}$, obtaining $\ln 0.18 = -0.00012t$ and $t = \dfrac{\ln 0.18}{-0.00012} \approx 14{,}290$, or approximately 14,290 years.

13. a. $f(t) = 157e^{-0.55t}$, so $f'(t) = 157(-0.55)e^{-0.55t} = -86.35e^{-0.55t}$. So the number of annual bank failures was changing at the rate of $f'(1) \approx -49.82$; that is, it was dropping at the rate of approximately 50/year in 2011.

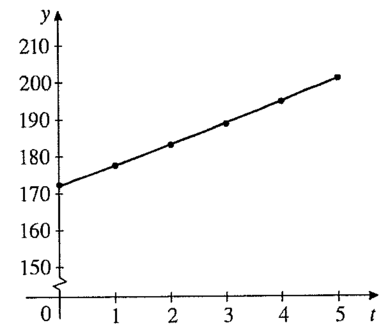
b. The projected number of failures in 2013 is $f(3) = 157e^{-0.55(3)} \approx 30.15$, or approximately 30.

14. $f(t) = 172.2e^{0.031t}$.

a. The projected number of online shoppers (in millions) from 2010 through 2015 is shown in the table.

t	0	1	2	3	4	5
$f(t)$	172.2	177.6	183.2	189.0	194.9	201.1

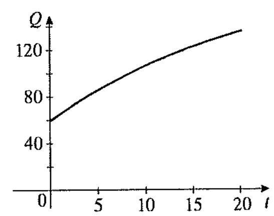
b.



15. a. $S = S_0e^{-kt}$, so $S(0) = S_0 = 100$. Thus, $S(t) = 100e^{kt}$. Next, $S(5) = 150$ gives $100e^{5k} = 150$, so $e^{5k} = \frac{150}{100} = 1.5$, $5k = \ln 1.5$, and $k \approx 0.0811$. Thus, the model is $S(t) = 100e^{0.0811t}$.

b. The sales of Garland Corporation in 2013 were $S(3) = 100e^{0.0811(3)} \approx 127.5$, or approximately \$127.5 million.

16.

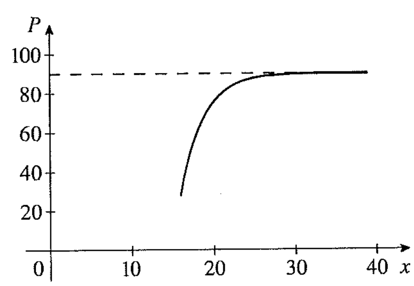


a. $Q(0) = 120(1 - e^0) + 60 = 60$, or 60 wpm.

b. $Q(10) = 120(1 - e^{-0.5}) + 60 = 107.22$, or approximately 107 wpm.

c. $Q(20) = 120(1 - e^{-1}) + 60 = 135.65$, or approximately 136 wpm.

17.



a. The percentage of 16-year-olds with a driver's license was $P(16) = 90\left[1 - e^{-0.37(16-15)}\right] \approx 27.8$, or approximately 27.8.

b. The percentage of 20-year-olds with a driver's license was $P(2) = 90\left[1 - e^{-0.37(20-15)}\right] \approx 75.8$, or approximately 75.8.

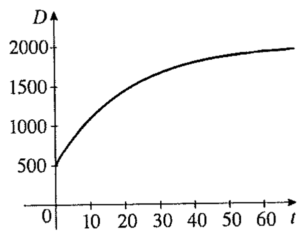
c. The percentage of 39-year-olds with a driver's license was $P(39) = 90\left[1 - e^{-0.37(39-15)}\right] \approx 90.0$, or approximately 90.0.

18. a. $S(t) = 50{,}000 + Ae^{-kt}$. Using the condition $S(1) = 83{,}515$ and $S(3) = 65{,}055$, we have $S(1) = 50{,}000 + Ae^{-k} = 83{,}515$ and $S(3) = 50{,}000 + Ae^{-3k} = 65{,}055$. The first equation gives $Ae^{-k} = 33{,}515$ and the second gives $Ae^{-3k} = 15055$, so $\dfrac{Ae^{-k}}{Ae^{-3k}} = \dfrac{33{,}515}{15{,}055}$, $e^{2k} = \dfrac{33{,}515}{15{,}055}$, and $k = \frac{1}{2}\ln\frac{33{,}515}{15{,}055} \approx 0.40014$.

b. $A = 33{,}515e^k = 33{,}515e^{0.40014} = 50{,}006$, so $S(t) = 50{,}000 + 50{,}006e^{-0.40014t}$. In particular, $S(4) = 50{,}000 + 50006e^{-0.40014(4)} \approx 60{,}090$, or approximately \$60,090.

c. $S'(t) = \dfrac{d}{dt}\left(50{,}000 + 50{,}006e^{-0.40014t}\right) = 50{,}006(-0.40014)e^{-0.40014t} = -20{,}009.4e^{-0.40014t}$, and so $S'(4) = -20{,}009.4e^{-0.40014(4)} \approx -4037.6$. That is, the sales volume is falling by approximately \$4038/week.

19.



a. After 1 month, the demand is $D(1) = 2000 - 1500e^{-0.05} \approx 573$, after 12 months it is $D(12) = 2000 - 1500e^{-0.6} \approx 1177$, after 24 months it is $D(24) = 2000 - 1500e^{-1.2} \approx 1548$, and after 60 months, it is $D(60) = 2000 - 1500e^{-3} \approx 1925$.

b. $\lim_{t\to\infty} D(t) = \lim_{t\to\infty} \left(2000 - 1500e^{-0.05t}\right) = 2000$, and we conclude that the demand is expected to stabilize at 2000 computers per month.

c. $D'(t) = -1500e^{-0.05t}(-0.05) = 75e^{-0.05t}$. Therefore, the rate of growth after 10 months is $D'(10) = 75e^{-0.5} \approx 45.49$, or approximately 46 computers per month.

20. a. The proportion that will fail after 3 years is $P(3) = 100\left(1 - e^{-0.3}\right) \approx 25.92\%$. Therefore, 74% will be usable.

b. $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} 100\left(1 - e^{-0.1t}\right) = 100$, so all will eventually fail, as one might expect.

21. a. The length is given by $f(5) = 200\left(1 - 0.956e^{-0.18\cdot 5}\right) \approx 122.26$, or approximately 122.3 cm.

b. $f'(t) = 200(-0.956)e^{-0.18t}(-0.18) = 34.416e^{-0.18t}$, so a 5-year-old is growing at the rate of $f'(5) = 34.416e^{-0.18(5)} \approx 13.9925$, or approximately 14 cm/yr.

c. The maximum length is given by $\lim_{t\to\infty} 200\left(1 - 0.956e^{-0.18t}\right) = 200$, or 200 cm.

22. a. $Q(1) = \dfrac{1000}{1 + 199e^{-0.8}} \approx 11.06$, or 11 children.

b. $Q(10) = \dfrac{1000}{1 + 199e^{-8}} \approx 937.4$, or 937 children.

c. $\lim_{t\to\infty} \dfrac{1000}{1 + 199e^{-0.8t}} = 1000$, or 1000 children.

23. a. $N(0) = \dfrac{400}{1 + 39} = 10$ flies.

b. $\lim_{t\to\infty} \dfrac{400}{1 + 39e^{-0.16t}} = 400$ flies.

c. $N(20) = \dfrac{400}{1 + 39e^{-0.16(20)}} \approx 154.5$, or 154 flies.

d. $N'(t) = \dfrac{d}{dt}\left[400\left(1 + 39e^{-0.16t}\right)^{-1}\right] = -400\left(1 + 39e^{-0.16t}\right)^{-2} \dfrac{d}{dt}\left(39e^{-0.16t}\right) = \dfrac{2496e^{-0.16t}}{\left(1 + 39e^{-0.16t}\right)^2}$, so

$N'(20) = \dfrac{2496e^{-0.16\cdot 20}}{\left(1 + 39e^{-0.16\cdot 20}\right)^2} \approx 15.17$, or approximately 15 fruit flies per day.

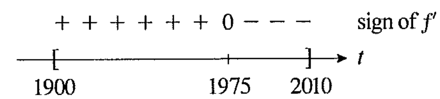
24. The projected population of citizens aged 45–64 in 2010 is $P(20) = \dfrac{197.9}{1 + 3.274e^{-0.0361(20)}} \approx 76.3962$, or 76.4 million.

25. $f(t) = \dfrac{40e^{-(t-1975)/20}}{\left[1+e^{-(t-1975)/20}\right]^2}$. Let $u = (t-1975)/20$. Then $\dfrac{du}{dt} = \dfrac{1}{20}$ and

$$f'(t) = f'(u)\frac{du}{dt} = 40 \cdot \frac{(1+e^{-u})^2(-1) - e^{-u}(2)(1+e^{-u})e^{-u}(-1)}{(1+e^{-u})^4}\left(\frac{1}{20}\right)$$

$$= 2 \cdot \frac{(1+e^{-u})e^{-u}\left[-(1+e^{-u})+2e^{-u}\right]}{(1+e^{-u})^4} = \frac{2e^{-u}(e^{-u}-1)}{(1+e^{-u})^3} = \frac{2e^{-(t-1975)/20}\left[e^{-(t-1975)/20}-1\right]}{\left[1+e^{-(t-1975)/20}\right]^3}.$$

Setting $f'(t) = 0$ gives $e^{-(t-1975)/20} = 1$, so $-(t-1975)/20 = \ln 1 = 0$ and $t = 1975$ is a critical number of f. From the sign diagram, we see that f has a relative maximum value at $t = 1975$. Since $t = 1975$ is the only critical number in the interval $(1900, 2010)$, we see that it gives an absolute maximum value of $f(1975) \approx 10$. We conclude that the maximum rate of production of crude oil in the U.S. occurred around 1975 and was approximately 10 million barrels per day.



26. The expected population of the U.S. in 2020 is $P(3) = \dfrac{616.5}{1+4.02e^{-0.5(3)}} \approx 324.99$, or approximately 325 million people.

27. The first of the given conditions implies that $f(0) = 300$, that is, $300 = \dfrac{3000}{1+Be^0} = \dfrac{3000}{1+B}$. Thus, $1+B = 10$, and $B = 9$. Therefore, $f(t) = \dfrac{3000}{1+9e^{-kt}}$. Next, the condition $f(2) = 600$ gives the equation $600 = \dfrac{3000}{1+9e^{-2k}}$, so $1+9e^{-2k} = 5$, $e^{-2k} = \frac{4}{9}$, and $k = -\frac{1}{2}\ln\frac{4}{9}$. Therefore, $f(t) = \dfrac{3000}{1+9e^{(1/2)t\cdot\ln(4/9)}} = \dfrac{3000}{1+9\left(\frac{4}{9}\right)^{t/2}}$. The number of students who had heard about the policy four hours later is given by $f(4) = \dfrac{3000}{1+9\left(\frac{4}{9}\right)^2} = 1080$, or 1080 students. To find the rate at which the rumor was spreading at any time time, we compute

$$f'(t) = \tfrac{d}{dt}\left[3000\left(1+9e^{-0.405465t}\right)^{-1}\right] = (3000)(-1)\left(1+9e^{-0.405465}\right)^{-2}\tfrac{d}{dt}\left(9e^{-0.405465t}\right)$$

$$= -3000(9)(-0.405465)e^{-0.405465t}\left(1+9e^{-0.405465t}\right)^{-2} = \frac{10947.555e^{-0.405465t}}{\left(1+9e^{-0.405465t}\right)^2}.$$

In particular, the rate at which the rumor was spreading 4 hours after the ceremony is given by

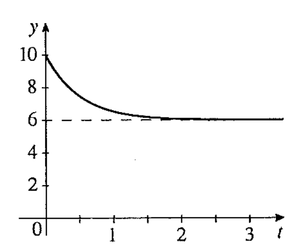
$f'(4) = \dfrac{10947.555e^{-0.405465\cdot 4}}{\left(1+9e^{-0.405465\cdot 4}\right)^2} \approx 280.26$. Thus, the rumor is spreading at the rate of 280 students per hour.

28. **a.** $f(t) = 6+4e^{-2t}$, so $f'(t) = -8e^{-2t} < 0$ for all t in $(0, \infty)$. Thus, f is decreasing on $(0, \infty)$.

b. $f''(t) = 16e^{-2t} > 0$ for all t in $(0, \infty)$, so f is concave upward on $(0, \infty)$.

c. $\lim\limits_{t\to\infty} f(t) = \lim\limits_{t\to\infty}\left(6+4e^{-2t}\right) = 6$.

d.



29. $x(t) = \dfrac{15\left(1-\left(\frac{2}{3}\right)^{3t}\right)}{1-\frac{1}{4}\left(\frac{2}{3}\right)^{3t}}$, so $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \dfrac{15\left[1-\left(\frac{2}{3}\right)^{3t}\right]}{1-\frac{1}{4}\left(\frac{2}{3}\right)^{3t}} = \dfrac{15(1-0)}{1-0} = 15$, or 15 lb.

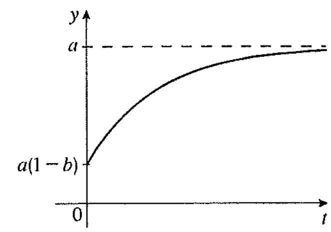
30. a. $f'(t) = \frac{d}{dt}\left[a\left(1-be^{-kt}\right)\right] = \frac{d}{dt}(a) - \frac{d}{dt}abe^{-kt} = 0 - be^{-kt}(-k) = bke^{-kt}$. Because $f'(t) > 0$ for all $t \geq 0$, f is increasing on $(0, \infty)$.

b. $f''(t) = \frac{d}{dt}\left(bke^{-kt}\right) = -bk^2e^{-kt} < 0$ on $(0, \infty)$, and the conclusion follows.

c. $\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[a\left(1-be^{-kt}\right)\right] = \lim_{t\to\infty} a - \lim_{t\to\infty} abe^{-kt}$

$= a - 0 = a$.

d.



31. a. $C(t) = \dfrac{k}{b-a}\left(e^{-at} - e^{-bt}\right)$, so

$$C'(t) = \frac{k}{b-a}\left(-ae^{-at} + be^{-bt}\right) = \frac{kb}{b-a}\left[e^{-bt} - \left(\frac{a}{b}\right)e^{-at}\right] = \frac{kb}{b-a}e^{-bt}\left[1 - \frac{a}{b}e^{(b-a)t}\right].$$

$C'(t) = 0$ implies that $1 = \dfrac{a}{b}e^{(b-a)t}$, or $t = \dfrac{\ln\left(\frac{b}{a}\right)}{b-a}$. The sign diagram of C' shows that this value of t gives a maximum.

+ + + + + 0 − − − − sign of C'

t: 0 ... $\dfrac{\ln \frac{b}{a}}{b-a}$

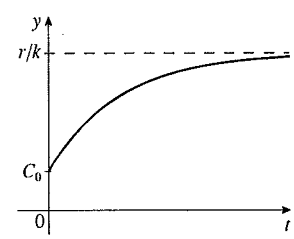
b. $\lim_{t\to\infty} C(t) = 0$.

32. a. $\lim_{t\to\infty}\left[\frac{r}{k} - \left(\frac{r}{k} - C_0\right)e^{-kt}\right] = \frac{r}{k}$, and this shows that in the long run the concentration of the glucose solution approaches $\frac{r}{k}$.

b. $C'(t) = -\left(\frac{r}{k} - C_0\right)e^{-kt}(-k) = k\left(\frac{r}{k} - C_0\right)e^{-kt} > 0$ for all $t > 0$ because $\frac{r}{k} > C_0$ for all $t > 0$. Thus, C is increasing on $(0, \infty)$.

c. $C''(t) = -k^2\left(\frac{r}{k} - C_0\right)e^{-kt} < 0$ because $\frac{r}{k} > C_0$ for all $t > 0$. Thus, the graph of C is concave downward.

d.



33. a. We solve $Q_0e^{-kt} = \frac{1}{2}Q_0$ for t, obtaining $e^{-kt} = \frac{1}{2}$, $\ln e^{-kt} = \ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$, $-kt = -\ln 2$, and so $\bar{t} = \frac{\ln 2}{k}$.

b. $\bar{t} = \frac{\ln 2}{0.0001238} \approx 5598.927$, or approximately 5599 years.

34. a. $Q'(t) = \frac{d}{dt}\left(\frac{A}{1+Be^{-kt}}\right) = A\frac{d}{dt}\left(1+Be^{-kt}\right)^{-1} = -A\left(1+Be^{-kt}\right)^{-2}\frac{d}{dt}\left(1+Be^{-kt}\right)$

$= -A\left(1+Be^{-kt}\right)^{-2}\left(-kBe^{-kt}\right) = \frac{kABe^{-kt}}{\left(1+Be^{-kt}\right)^2}$ (1)

Next,

$kQ\left(1-\frac{Q}{A}\right) = k\left(\frac{A}{1+Be^{-kt}}\right)\left(1-\frac{1}{1+Be^{-kt}}\right) = k\left(\frac{A}{1+Be^{-kt}}\right)\left(\frac{1+Be^{-kt}-1}{1+Be^{-kt}}\right) = \frac{kABe^{-kt}}{\left(1+Be^{-kt}\right)^2}$ (2).

The desired result follows by comparing equations (1) and (2).

b. Because $Q(t) < A$, $Q' = kQ\left(1-\frac{Q}{A}\right) > 0$, and we see that Q is increasing on $(0, \infty)$.

35. a. From the results of Exercise 34, we have $Q' = kQ\left(1-\frac{Q}{A}\right)$, so

$Q'' = \frac{d}{dt}\left(kQ - \frac{k}{A}Q^2\right) = kQ' - \frac{2k}{A}QQ' = \frac{k}{A}Q'(A-2Q)$. Setting $Q''=0$ gives $Q = \frac{A}{2}$ since $Q' > 0$ for all t. Furthermore, $Q'' > 0$ if $Q < \frac{A}{2}$ and $Q'' < 0$ if $Q > \frac{A}{2}$, So the graph of Q has an inflection point when $Q = \frac{A}{2}$. To find the value of t, we solve the equation $\frac{A}{2} = \frac{A}{1+Be^{-kt}}$, obtaining $1+Be^{-kt} = 2$, $Be^{-kt} = 1$, $e^{-kt} = \frac{1}{B}$, $-kt = \ln\frac{1}{B} = -\ln B$, and so $t = \frac{\ln B}{k}$.

b. The quantity Q increases most rapidly at the instant of time when it reaches one-half of the maximum quantity. This occurs at $t = \frac{\ln B}{k}$.

36. $Q(t) = \frac{A}{1+Be^{-kt}}$, so $Q(t_1) = \frac{A}{1+Be^{-kt_1}} = Q_1$ implies that $A = Q_1 + Q_1e^{-kt_1}B$, so $e^{-kt_1} = \frac{A-Q_1}{BQ_1}$ (1).

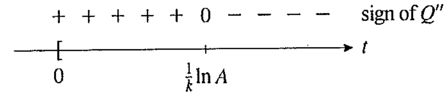
Next, we have $Q(t_2) = \frac{A}{1+Be^{-kt_2}} = Q_2$, and this leads to $e^{-kt_2} = \frac{A-Q_2}{BQ_2}$ (2). Dividing equation (1) by equation (2) gives $\frac{e^{-kt_1}}{e^{-kt_2}} = \frac{A-Q_1}{BQ_1}\cdot\frac{BQ_2}{A-Q_2}$, so $e^{k(t_2-t_1)} = \frac{Q_2(A-Q_1)}{Q_1(A-Q_2)}$, $k(t_2-t_1) = \ln\frac{Q_2(A-Q_1)}{Q_1(A-Q_2)}$, and $k = \frac{1}{t_2-t_1}\ln\frac{Q_2(A-Q_1)}{Q_1(A-Q_2)}$.

37. We use the result of Exercise 36 with $t_1 = 14$, $t_2 = 21$, $A = 600$, $Q_1 = 76$, and $Q_2 = 167$ to obtain

$k = \frac{1}{21-14}\ln\left[\frac{167(600-76)}{76(600-167)}\right] \approx 0.14.$

38. a. $Q'(t) = Ce^{-Ae^{-kt}}\frac{d}{dt}\left(-Ae^{-kt}\right) = -ACe^{-Ae^{-kt}}\cdot e^{-kt}(-k) = ACke^{\left(-Ae^{-kt}-kt\right)}$.

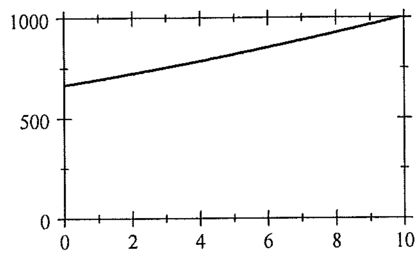
b. $Q''(t) = ACke^{\left(-Ae^{-kt}-kt\right)}\left[-k - Ae^{-kt}(-k)\right] = 0$, if $Ae^{-kt} = 1$, so $e^{-kt} = \frac{1}{A}$, $-kt = \ln\frac{1}{A}$, and $t = -\frac{1}{k}\ln\frac{1}{A} = \frac{1}{k}\ln A$. The sign diagram shows that $t = \frac{1}{k}\ln A$ is an inflection point, and so the growth is most rapid at this time.



c. $\lim_{t\to\infty} Q(t) = C$.

Using Technology page 403

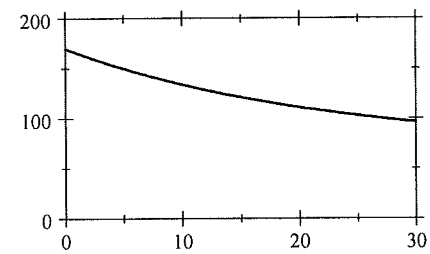
1. a.



b. $T(0) = 666$ million; $T(8) \approx 926.8$ million.

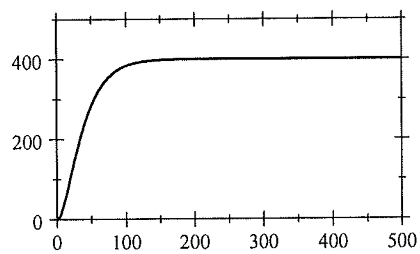
c. $T'(8) \approx 38.3$ million/yr/yr.

2. a.



b. $T(t) = 120$ when $t \approx 15.54$ min.

3. a.

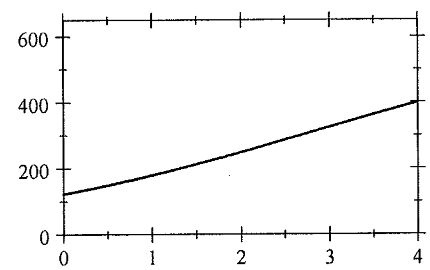


b. $\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \left[-20(t+20)e^{-0.05t} + 400\right]$

$= 400,$

so Starr will eventually sell 400,000 copies of Laser Beams.

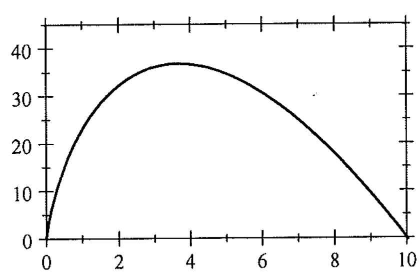
4. a.



b. $P(3) \approx 325$ million.

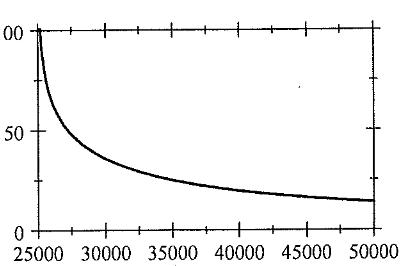
c. $P'(3) \approx 76.84$ million per 30 years.

5. a.



b. $R'(x) = 0$ when $x \approx 3.68$.

6. a.

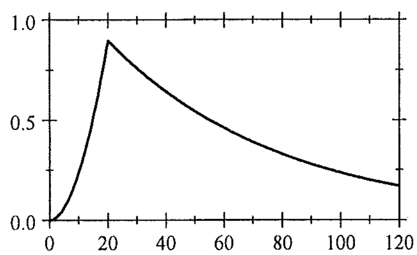


b. $f(x) = 25$ when $x \approx \$35{,}038.78$/yr.

c. $\lim_{x\to 25{,}000^+} f(x) = \infty$. If Christine withdraws \$25,000/yr she will be withdrawing only the interest, and so the account will never be depleted.

d. $\lim_{x\to\infty} f(x) = 0$. If Christine withdraws everything in her account, it is depleted immediately.

7. a.



b. The initial concentration is 0.

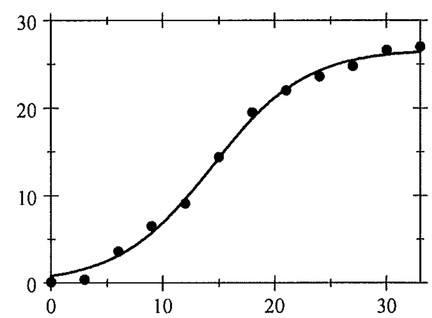
c. $C(10) \approx 0.237 \text{ g/cm}^3$.

d. $C(30) \approx 0.760 \text{ g/cm}^3$.

e. $\lim_{t \to \infty} C(t) = 0$.

8. a. $f(t) = \dfrac{26.71}{1 + 31.74e^{-0.24t}}$.

b.



c. At midnight, the snowfall was accumulating at the rate of $f'(12) \approx 1.476$, or approximately 1.5 in/hr. At noon on February 7, it was accumulating at the rate of $f'(24) \approx 0.530$, or approximately 0.5 in/hr.

d. The inflection point is approximately (14.4, 13.4), so snow was accumulating at the greatest rate at about 2:24 A.M. on February 7. The rate of accumulation was $f'(14.4) \approx 1.60$, or approximately 1.6 in/hr.

CHAPTER 5 Concept Review Questions page 405

1. power, 0, 1, exponential

2. a. $(-\infty, \infty)$, $(0, \infty)$ **b.** $(0, 1)$, $(-\infty, \infty)$

3. a. $(0, \infty)$, $(-\infty, \infty)$, $(1, 0)$ **b.** < 1, > 1

4. a. x **b.** x

5. accumulated amount, principal, nominal interest rate, number of conversion periods, term

6. $\left(1 + \dfrac{r}{m}\right)^m - 1$

7. Pe^{rt}

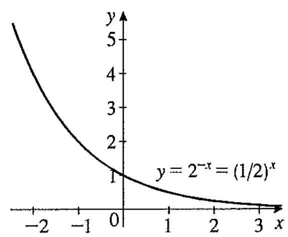
8. a. $e^{f(x)} \cdot f'(x)$ **b.** $\dfrac{f'(x)}{f(x)}$

9. a. initially, growth **b.** decay **c.** time, one-half

10. a. horizontal asymptote, C **b.** horizontal asymptote, A, carrying capacity

CHAPTER 5 Review Exercises page 406

1.



$\left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x}$, so the two graphs are the same.

2. If $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$, then $\log_{2/3}\left(\frac{27}{8}\right) = -3$.

3. $16^{-3/4} = 0.125$ is equivalent to $-\frac{3}{4} = \log_{16} 0.125$.

4. $\log_4(2x+1) = 2$, $(2x+1) = 4^2 = 16$, $2x = 15$, and so $x = \frac{15}{2}$.

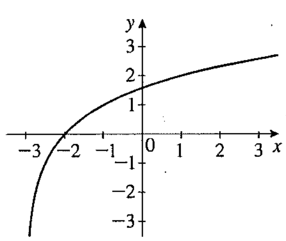
5. $\ln(x-1) + \ln 4 = \ln(2x+4) - \ln 2$, so $\ln(x-1) - \ln(2x+4) = -\ln 2 - \ln 4 = -(\ln 2 + \ln 4)$, $\ln\left(\frac{x-1}{2x+4}\right) = -\ln 8 = \ln\frac{1}{8}$, $\left(\frac{x-1}{2x+4}\right) = \frac{1}{8}$, $8x - 8 = 2x + 4$, $6x = 12$, and so $x = 2$. Check: LHS $= \ln(2-1) + \ln 4 = \ln 4$; RHS $= \ln(4+4) - \ln 2 = \ln 8 - \ln 2 = \ln\frac{8}{2} = \ln 4$.

6. $\ln 30 = \ln(2 \cdot 3 \cdot 5) = \ln 2 + \ln 3 + \ln 5 = x + y + z$.

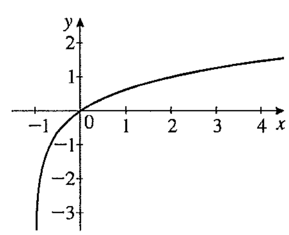
7. $\ln 3.6 = \ln\frac{36}{10} = \ln 36 - \ln 10 = \ln 6^2 - \ln(2 \cdot 5) = 2\ln 6 - \ln 2 - \ln 5 = 2(\ln 2 + \ln 3) - \ln 2 - \ln 5$
$= 2(x+y) - x - z = x + 2y - z$.

8. $\ln 75 = \ln\left(3 \cdot 5^2\right) = \ln 3 + 2\ln 5 = y + 2z$.

9. We first sketch the graph of $y = 2^{x+3}$, then reflect this graph with respect to the line $y = x + 3$.



10. We first sketch the graph of $y = 3^{x+1}$, then reflect this graph with respect to the line $y = x + 1$.



11. **a.** Using Formula (6) with $P = 10{,}000$, $r = 0.06$, $m = 365$ and $t = 2$, we have
$$A = 10{,}000\left(1 + \frac{0.06}{365}\right)^{365(2)} = 11{,}274.86, \text{ or } \$11{,}274.86.$$

b. Using Formula (10) with $P = 10{,}000$, $r = 0.06$, and $t = 2$, we have $A = 10{,}000e^{0.06(2)} = 11{,}274.97$, or \$11,274.97.

12. Using Formula (6), with $A = 10{,}000$, $P = 12{,}000$, $m = 4$, and $t = 3$, we have $A = 10{,}000\left(1 + \frac{r}{4}\right)^{4(3)} = 12{,}000$, or $\left(1 + \frac{r}{4}\right)^{12} = 1.2$. Solving for r, we have $\frac{r}{4} = (1.2)^{1/12} - 1$, so $r = 4\left[(1.2)^{1/12} - 1\right] \approx 0.0612$, or 6.12% per year.

13. Using Formula (6) with $A = 10{,}000$, $P = 15{,}000$, $r = 0.06$, and $m = 4$, we have $A = 10{,}000\left(1 + \frac{0.06}{4}\right)^{4t} = 15{,}000$, or $\left(1 + \frac{0.06}{4}\right)^{4t} = 1.5$. Solving for t, we have $4t \ln(1.015) = \ln 1.5$, so $t = \dfrac{\ln 1.5}{4 \ln 1.015} \approx 6.808$, or approximately 6.8 years.

14. Using Formula (7) to compute the effective rate of interest with $r = 0.08$ and $m = 4$, we have $r_{\text{eff}} = \left(1 + \dfrac{r}{m}\right)^m - 1$, or $0.08 = \left(1 + \dfrac{r}{4}\right)^4 - 1$. Solving for r, we have $\left(1 + \dfrac{r}{4}\right)^4 = 1.08$, $\dfrac{r}{4} = 1.08^{1/4} - 1$, and so $r = 4\left[(1.08)^{1/4} - 1\right] \approx 0.0777$, or approximately 7.77%/yr.

15. $f(x) = xe^{2x}$, so $f'(x) = e^{2x} + xe^{2x}(2) = (1 + 2x)e^{2x}$.

16. $f(t) = \sqrt{t}e^t + t$, so $f'(t) = \frac{1}{2}t^{-1/2}e^t + t^{1/2}e^t + 1 = \dfrac{e^t}{2\sqrt{t}} + \sqrt{t}e^t + 1$.

17. $g(t) = \sqrt{t}e^{-2t}$, so $g'(t) = \frac{1}{2}t^{-1/2}e^{-2t} + \sqrt{t}e^{-2t}(-2) = \dfrac{1 - 4t}{2\sqrt{t}e^{2t}}$.

18. $g(x) = e^x\left(1 + x^2\right)^{1/2}$, so

$$g'(x) = e^x \tfrac{d}{dx}\left(1 + x^2\right)^{1/2} + \left(1 + x^2\right)^{1/2} \tfrac{d}{dx}e^x = e^x \tfrac{1}{2}\left(1 + x^2\right)^{-1/2}(2x) + \left(1 + x^2\right)^{1/2} e^x$$
$$= e^x\left(1 + x^2\right)^{-1/2}\left(x + 1 + x^2\right) = \frac{e^x\left(x^2 + x + 1\right)}{\sqrt{1 + x^2}}.$$

19. $y = \dfrac{e^{2x}}{1 + e^{-2x}}$, so $y' = \dfrac{\left(1 + e^{-2x}\right)e^{2x}(2) - e^{2x} \cdot e^{-2x}(-2)}{\left(1 + e^{-2x}\right)^2} = \dfrac{2\left(e^{2x} + 2\right)}{\left(1 + e^{-2x}\right)^2}$.

20. $f(x) = e^{2x^2 - 1}$, so $f'(x) = e^{2x^2 - 1}(4x) = 4xe^{2x^2 - 1}$.

21. $f(x) = xe^{-x^2}$, so $f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = \left(1 - 2x^2\right)e^{-x^2}$.

22. $g(x) = \left(1 + e^{2x}\right)^{3/2}$, so $g'(x) = \frac{3}{2}\left(1 + e^{2x}\right)^{1/2} \cdot e^{2x}(2) = 3e^{2x}\left(1 + e^{2x}\right)^{1/2}$.

23. $f(x) = x^2e^x + e^x$, so $f'(x) = 2xe^x + x^2e^x + e^x = \left(x^2 + 2x + 1\right)e^x = (x + 1)^2 e^x$.

24. $g(t) = t \ln t$, so $g'(t) = \ln t + t\left(\frac{1}{t}\right) = \ln t + 1$.

25. $f(x) = \ln\left(e^{x^2} + 1\right)$, so $f'(x) = \dfrac{e^{x^2}(2x)}{e^{x^2} + 1} = \dfrac{2xe^{x^2}}{e^{x^2} + 1}$.

26. $f(x) = \dfrac{x}{\ln x}$, so $f'(x) = \dfrac{\ln x \dfrac{d}{dx}x - x\dfrac{d}{dx}\ln x}{(\ln x)^2} = \dfrac{\ln x - x \cdot \dfrac{1}{x}}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2}$.

27. $f(x) = \dfrac{\ln x}{x + 1}$, so $f'(x) = \dfrac{(x + 1)\left(\frac{1}{x}\right) - \ln x}{(x + 1)^2} = \dfrac{1 + \frac{1}{x} - \ln x}{(x + 1)^2} = \dfrac{x - x\ln x + 1}{x(x + 1)^2}$.

28. $y = (x + 1)e^x$, so $y' = e^x + (x + 1)e^x = (x + 2)e^x$.

29. $y = \ln\left(e^{4x} + 3\right)$, so $y' = \dfrac{e^{4x}(4)}{e^{4x}+3} = \dfrac{4e^{4x}}{e^{4x}+3}$.

30. $f(r) = \dfrac{re^r}{1+r^2}$, so $f'(r) = \dfrac{\left(1+r^2\right)\left(e^r + re^r\right) - re^r(2r)}{\left(1+r^2\right)^2} = \dfrac{\left(r^3 - r^2 + r + 1\right)e^r}{\left(1+r^2\right)^2}$.

31. $f(x) = \dfrac{\ln x}{1+e^x}$, so

$$f'(x) = \frac{(1+e^x)\dfrac{d}{dx}\ln x - \ln x\dfrac{d}{dx}(1+e^x)}{(1+e^x)^2} = \frac{(1+e^x)\left(\dfrac{1}{x}\right) - (\ln x)e^x}{(1+e^x)^2} = \frac{1+e^x - xe^x\ln x}{x(1+e^x)^2}$$
$$= \frac{1+e^x(1-x\ln x)}{x(1+e^x)^2}.$$

32. $g(x) = \dfrac{e^{x^2}}{1+\ln x}$, so $g'(x) = \dfrac{(1+\ln x)e^{x^2}(2x) - e^{x^2}\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \dfrac{\left(2x^2 + 2x^2\ln x - 1\right)e^{x^2}}{x(1+\ln x)^2}$.

33. $y = \ln(3x+1)$, so $y' = \dfrac{3}{3x+1}$ and $y'' = 3\dfrac{d}{dx}(3x+1)^{-1} = -3(3x+1)^{-2}(3) = -\dfrac{9}{(3x+1)^2}$.

34. $y = x\ln x$, so $y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$ and $y'' = \frac{1}{x}$.

35. $h'(x) = g'(f(x))f'(x)$. But $g'(x) = 1 - \dfrac{1}{x^2}$ and $f'(x) = e^x$, so $f(0) = e^0 = 1$ and $f'(0) = e^0 = 1$. Therefore, $h'(0) = g'(f(0))f'(0) = g'(1)f'(0) = 0 \cdot 1 = 0$.

36. $h'(1) = g'(f(1))f'(1)$ by the Chain Rule. Now $g'(x) = \dfrac{(x-1)-(x+1)}{(x-1)^2} = -\dfrac{2}{(x-1)^2}$, $f'(x) = \dfrac{1}{x}$, and $f(1) = 0$, so $h'(1) = -\dfrac{2}{(-1)^2} \cdot 1 = -2$.

37. $y = \left(2x^3+1\right)\left(x^2+2\right)^3$, so $\ln y = \ln\left(2x^3+1\right) + 3\ln\left(x^2+2\right)$,

$$\frac{y'}{y} = \frac{6x^2}{2x^3+1} + \frac{3(2x)}{x^2+2} = \frac{6x^2\left(x^2+2\right) + 6x\left(2x^3+1\right)}{\left(2x^3+1\right)\left(x^2+2\right)} = \frac{6x^4 + 12x^2 + 12x^4 + 6x}{\left(2x^3+1\right)\left(x^2+2\right)}$$
$$= \frac{18x^4 + 12x^2 + 6x}{\left(2x^3+1\right)\left(x^2+2\right)},$$

and so $y' = 6x\left(3x^3 + 2x + 1\right)\left(x^2+2\right)^2$.

38. $f(x) = \dfrac{x\left(x^2-2\right)^2}{x-1}$, so $\ln f(x) = \ln x + 2\ln\left(x^2-2\right) - \ln(x-1)$. Thus,

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{2(2x)}{x^2-2} - \frac{1}{x-1} = \frac{\left(x^2-2\right)(x-1) + 4x^2(x-1) - x\left(x^2-2\right)}{x(x-1)\left(x^2-2\right)} = \frac{4x^3 - 5x^2 + 2}{x(x-1)\left(x^2-2\right)},$$ and so

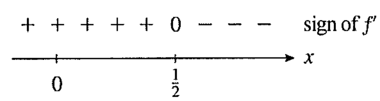
$$f'(x) = \frac{4x^3 - 5x^2 + 2}{x(x-1)\left(x^2-2\right)}\,\frac{x\left(x^2-2\right)^2}{x-1} = \frac{\left(4x^3 - 5x^2 + 2\right)\left(x^2-2\right)}{(x-1)^2}.$$

39. $y = e^{-2x}$, so $y' = -2e^{-2x}$. This gives the slope of the tangent line to the graph of $y = e^{-2x}$ at any point (x, y). In particular, the slope of the tangent line at $(1, e^{-2})$ is $y'(1) = -2e^{-2}$. The required equation is $y - e^{-2} = -2e^{-2}(x - 1)$, or $y = \frac{1}{e^2}(-2x + 3)$.

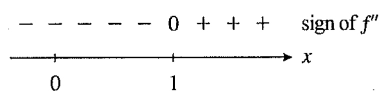
40. $y = xe^{-x}$, so $y' = e^{-x} + xe^{-x}(-1) = (1 - x)e^{-x}$. The slope of the tangent line at $(1, e^{-1})$ is 0. Therefore, an equation of the tangent line is $y = 1/e$.

41. $f(x) = xe^{-2x}$. We first gather the following information on f.

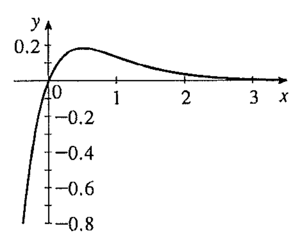
1. The domain of f is $(-\infty, \infty)$.
2. Setting $x = 0$ gives 0 as the y-intercept.
3. $\lim_{x \to -\infty} xe^{-2x} = -\infty$ and $\lim_{x \to \infty} xe^{-2x} = 0$.
4. The results of part 3 show that $y = 0$ is a horizontal asymptote.
5. $f'(x) = e^{-2x} + xe^{-2x}(-2) = (1 - 2x)e^{-2x}$. Observe that $f'(x) = 0$ at $x = \frac{1}{2}$, a critical point of f. The sign diagram of f' shows that f is increasing on $\left(-\infty, \frac{1}{2}\right)$ and decreasing on $\left(\frac{1}{2}, \infty\right)$.



6. The results of part 5 show that $\left(\frac{1}{2}, \frac{1}{2}e^{-1}\right)$ is a relative maximum.
7. $f''(x) = -2e^{-2x} + (1 - 2x)e^{-2x}(-2) = 4(x - 1)e^{-2x} = 0$ if $x = 1$. The sign diagram of f'' shows that the graph of f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.



8. f has an inflection point at $(1, 1/e^2)$.



42. $f(x) = x^2 - \ln x$. We first gather the following information on f.

1. The domain of f is $(0, \infty)$.
2. There is no y-intercept.
3. $\lim_{x \to \infty} (x^2 - \ln x) = \infty$.
4. There is no asymptote.
5. $f'(x) = 2x - \frac{1}{x} = \frac{2x^2 - 1}{x}$. Setting $f'(x) = 0$ gives $x = \pm\frac{\sqrt{2}}{2}$. We reject the negative root, so $x = \frac{\sqrt{2}}{2}$ is a critical point of f. The sign diagram of f' shows that f is decreasing on $\left(0, \frac{\sqrt{2}}{2}\right)$ and increasing on $\left(\frac{\sqrt{2}}{2}, \infty\right)$.