b. At the beginning of June, the population of aphids is changing at the rate of $F'(1) \approx 226.02$; that is, it is increasing at the rate of 226 aphids on a typical bean stem per month. At the beginning of July, the population is changing at the rate of $F'(2) \approx -238.3$; that is, it is decreasing at the rate of 238 aphids per month.

5.5 Differentiation of Logarithmic Functions

Concept Questions

page 386

1. a. $f'(x) = \frac{1}{x}$.

b. $g'(x) = \frac{f'(x)}{f(x)}$.

2. See the procedure given on page 385 of the text.

Exercises

page 387

1. $f(x) = 5\ln x$, so $f'(x) = 5\left(\frac{1}{x}\right) = \frac{5}{x}$.

2. $f(x) = \ln 5x$, so $f'(x) = \frac{5}{5x} = \frac{1}{x}$.

3. $f(x) = \ln(x+1)$, so $f'(x) = \frac{1}{x+1}$.

4. $g(x) = \ln(2x+1)$, so $g'(x) = \frac{2}{2x+1}$.

5. $f(x) = \ln x^8$, so $f'(x) = \frac{8x^7}{x^8} = \frac{8}{x}$.

6. $h(t) = 2\ln t^5$, so $h'(t) = \frac{2}{t^5} \cdot 5t^4 = \frac{10}{t}$.

7. $f(x) = \ln x^{1/2}$, so $f'(x) = \frac{\frac{1}{2}x^{-1/2}}{x^{1/2}} = \frac{1}{2x}$.

8. $f(x) = \ln\left(x^{1/2}+1\right)$, so $f'(x) = \frac{\frac{1}{2}x^{-1/2}}{x^{1/2}+1} = \frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)}$.

9. $f(x) = \ln\left(\frac{1}{x^2}\right) = \ln x^{-2}$, so $f'(x) = -\frac{2x^{-3}}{x^{-2}} = -\frac{2}{x}$.

10. $f(x) = \ln\frac{1}{2x^3} = \ln 1 - \ln\left(2x^3\right) = -\ln 2 - \ln x^3$, so $f'(x) = -\frac{3x^2}{x^3} = -\frac{3}{x}$.

11. $f(x) = \ln\left(4x^2-5x+3\right)$, so $f'(x) = \frac{8x-5}{4x^2-5x+3} = \frac{8x-5}{4x^2-5x+3}$.

12. $f(x) = \ln\left(3x^2-2x+1\right)$, so $f'(x) = \frac{6x-2}{3x^2-2x+1} = \frac{2(3x-1)}{3x^2-2x+1}$.

13. $f(x) = \ln\left(\frac{2x}{x+1}\right) = \ln 2x - \ln(x+1)$, so

$$f'(x) = \frac{2}{2x} - \frac{1}{x+1} = \frac{1}{x} - \frac{1}{x+1} = \frac{(x+1)-x}{x(x+1)} = \frac{x+1-x}{x(x+1)} = \frac{1}{x(x+1)}.$$

14. $f(x) = \ln(x+1) - \ln(x-1)$, so $f'(x) = \frac{1}{x+1} - \frac{1}{x-1} = \frac{(x-1)-(x+1)}{x^2-1} = -\frac{2}{x^2-1}$.

15. $f(x) = x^2\ln x$, so $f'(x) = x^2\left(\frac{1}{x}\right) + (\ln x)(2x) = x + 2x\ln x = x(1+2\ln x)$.

16. $f(x) = 3x^2 \ln 2x$, so $f'(x) = 6x \ln 2x + 3x^2 \cdot \frac{2}{2x} = 6x \ln 2x + 3x = 3x(2 \ln 2x + 1)$.

17. $f(x) = \frac{2 \ln x}{x}$, so $f'(x) = \frac{x\left(\frac{2}{x}\right) - 2 \ln x}{x^2} = \frac{2(1 - \ln x)}{x^2}$.

18. $f(x) = \frac{3 \ln x}{x^2}$, so $f'(x) = \frac{x^2\left(\frac{3}{x}\right) - (3 \ln x)(2x)}{x^4} = \frac{3x(1 - 2 \ln x)}{x^4} = \frac{3(1 - 2 \ln x)}{x^3}$.

19. $f(u) = \ln(u - 2)^3$, so $f'(u) = \frac{3(u-2)^2}{(u-2)^3} = \frac{3}{u-2}$.

20. $f(x) = \ln\left(x^3 - 3\right)^4$, so $f'(x) = \frac{4\left(x^3 - 3\right)^3\left(3x^2\right)}{\left(x^3 - 3\right)^4} = \frac{4\left(3x^2\right)}{x^3 - 3} = \frac{12x^2}{x^3 - 3}$.

21. $f(x) = (\ln x)^{1/2}$, so $f'(x) = \frac{1}{2}(\ln x)^{-1/2}\left(\frac{1}{x}\right) = \frac{1}{2x\sqrt{\ln x}}$.

22. $f(x) = (\ln x + x)^{1/2}$, so $f'(x) = \frac{1}{2}(\ln x + x)^{-1/2}\left(\frac{1}{x} + 1\right) = \frac{x+1}{2x\sqrt{\ln x + x}}$.

23. $f(x) = (\ln x)^2$, so $f'(x) = 2(\ln x)\left(\frac{1}{x}\right) = \frac{2 \ln x}{x}$.

24. $f(x) = 2(\ln x)^{3/2}$, so $f'(x) = 2\left(\frac{3}{2}\right)(\ln x)^{1/2}\left(\frac{1}{x}\right) = \frac{3(\ln x)^{1/2}}{x}$.

25. $f(x) = \ln\left(x^3 + 1\right)$, so $f'(x) = \frac{3x^2}{x^3 + 1}$.

26. $f(x) = \ln\left(x^2 - 4\right)^{1/2}$, so $f'(x) = \frac{\frac{1}{2}\left(x^2 - 4\right)^{-1/2}(2x)}{\left(x^2 - 4\right)^{1/2}} = \frac{x}{x^2 - 4}$.

27. $f(x) = e^x \ln x$, so $f'(x) = e^x \ln x + e^x\left(\frac{1}{x}\right) = \frac{e^x(x \ln x + 1)}{x}$.

28. $f(x) = e^x \ln\sqrt{x+3} = \frac{1}{2}e^x \ln(x+3)$, so $f'(x) = \frac{1}{2}\left[e^x \ln(x+3) + e^x \cdot \frac{1}{x+3}\right] = \frac{e^x[(x+3)\ln(x+3) + 1]}{2(x+3)}$.

29. $f(t) = e^{2t} \ln(t+1)$, so $f'(t) = e^{2t}\left(\frac{1}{t+1}\right) + \ln(t+1) \cdot \left(2e^{2t}\right) = \frac{[2(t+1)\ln(t+1) + 1]e^{2t}}{t+1}$.

30. $g(t) = t^2 \ln\left(e^{2t} + 1\right)$, so $g'(t) = 2t \ln\left(e^{2t} + 1\right) + t^2\left(\frac{2e^{2t}}{e^{2t} + 1}\right) = \frac{2t\left[\left(e^{2t} + 1\right)\ln\left(e^{2t} + 1\right) + te^{2t}\right]}{e^{2t} + 1}$.

31. $f(x) \frac{\ln x}{x^2}$, so $f'(x) = \frac{x^2\left(\frac{1}{x}\right) - \ln x(2x)}{x^4} = \frac{1 - 2 \ln x}{x^3}$.

32. $g(t) = \frac{t}{\ln t}$, so $g'(t) = \frac{(\ln t)(1) - t\left(\frac{1}{t}\right)}{(\ln t)^2} = \frac{\ln t - 1}{(\ln t)^2}$.

33. $f'(x) = \frac{d}{dx}[\ln(\ln x)] = \frac{\frac{d}{dx}(\ln x)}{\ln x} = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x\ln x}$.

34. $g'(x) = \frac{d}{dx}\left[\ln(e^x + \ln x)\right] = \frac{\frac{d}{dx}(e^x + \ln x)}{e^x + \ln x} = \frac{e^x + \frac{1}{x}}{e^x + \ln x} = \frac{xe^x + 1}{x(e^x + \ln x)}$.

35. $f(x) = \ln 2 + \ln x$, so $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$.

36. $f(x) = \ln(x+5)$, so $f'(x) = \frac{1}{x+5}$ and $f''(x) = \frac{d}{dx}(x+5)^{-1} = -(x+5)^{-2} = -\frac{1}{(x+5)^2}$.

37. $f(x) = \ln\left(x^2+2\right)$, so $f'(x) = \frac{2x}{\left(x^2+2\right)}$ and $f''(x) = \frac{\left(x^2+2\right)(2) - 2x(2x)}{\left(x^2+2\right)^2} = \frac{2\left(2-x^2\right)}{\left(x^2+2\right)^2}$.

38. $f(x) = (\ln x)^2$, so $f'(x) = 2(\ln x)\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$ and $f''(x) = \frac{x\left(\frac{2}{x}\right) - 2\ln x}{x^2} = \frac{2(1-\ln x)}{x^2}$.

39. $f'(x) = \frac{d}{dx}\left(x^2\ln x\right) = \frac{d}{dx}\left(x^2\right)\ln x + \frac{d}{dx}(\ln x)x^2 = 2x\ln x + \frac{1}{x}\cdot x^2 = 2x\ln x + x = x(2\ln x + 1)$ and

$f''(x) = \frac{d}{dx}[x(2\ln x + 1)] = \frac{d}{dx}(x)(2\ln x + 1) + \frac{d}{dx}(2\ln x + 1)x = 2\ln x + 1 + \frac{2}{x}\cdot x = 2\ln x + 3$.

40. $g(x) = e^{2x}\ln x$, so $g'(x) = e^{2x}\frac{d}{dx}\ln x + (\ln x)\frac{d}{dx}e^{2x} = \frac{e^{2x}}{x} + 2e^{2x}\ln x$ and

$g''(x) = \frac{2xe^{2x} - e^{2x}}{x^2} + \frac{2e^{2x}}{x} + 4e^{2x}\ln x = \frac{2xe^{2x} - e^{2x} + 2xe^{2x} + 4x^2e^{2x}\ln x}{x^2} = \frac{\left(4x - 1 + 4x^2\ln x\right)e^{2x}}{x^2}$.

41. $y = (x+1)^2(x+2)^3$, so

$\ln y = \ln(x+1)^2(x+2)^3 = \ln(x+1)^2 + \ln(x+2)^3 = 2\ln(x+1) + 3\ln(x+2)$.

Thus, $\frac{y'}{y} = \frac{2}{x+1} + \frac{3}{x+2} = \frac{2(x+2) + 3(x+1)}{(x+1)(x+2)} = \frac{5x+7}{(x+1)(x+2)}$ and

$y' = \frac{(5x+7)(x+1)^2(x+2)^3}{(x+1)(x+2)} = (5x+7)(x+1)(x+2)^2$.

42. $y = (3x+2)^4(5x-1)^2$, so $\ln y = 4\ln(3x+2) + 2\ln(5x-1)$. Thus,

$\frac{dy}{dx}\cdot\frac{1}{y} = \frac{4(3)}{3x+2} + \frac{2(5)}{5x-1} = \frac{12(5x-1) + 10(3x+2)}{(3x+2)(5x-1)} = \frac{60x - 12 + 30x + 20}{(3x+2)(5x-1)} = \frac{2(45x+4)}{(3x+2)(5x-1)}$, and so

$\frac{dy}{dx} = \frac{2(3x+2)^4(5x-1)^2(45x+4)}{(3x+2)(5x-1)} = 2(3x+2)^3(5x-1)(45x+4)$.

43. $y = (x-1)^2(x+1)^3(x+3)^4$, so $\ln y = 2\ln(x-1) + 3\ln(x+1) + 4\ln(x+3)$. Thus,

$\frac{y'}{y} = \frac{2}{x-1} + \frac{3}{x+1} + \frac{4}{x+3} = \frac{2(x+1)(x+3) + 3(x-1)(x+3) + 4(x-1)(x+1)}{(x-1)(x+1)(x+3)}$

$= \frac{2x^2 + 8x + 6 + 3x^2 + 6x - 9 + 4x^2 - 4}{(x-1)(x+1)(x+3)} = \frac{9x^2 + 14x - 7}{(x-1)(x+1)(x+3)}$, and so

$y' = \frac{9x^2 + 14x - 7}{(x-1)(x+1)(x+3)}\cdot y = \frac{\left(9x^2 + 14x - 7\right)(x-1)^2(x+1)^3(x+3)^4}{(x-1)(x+1)(x+3)}$

$= \left(9x^2 + 14x - 7\right)(x-1)(x+1)^2(x+3)^3$.

44. $y = (3x+5)^{1/2}(2x-3)^4$, so $\ln y = \frac{1}{2}\ln(3x+5) + 4\ln(2x-3)$. Thus,

$$\frac{y'}{y} = \frac{1}{2}\cdot\frac{1(3)}{3x+5} + 4\cdot\frac{2}{2x-3} = \frac{3}{2(3x+5)} + \frac{8}{2x-3} = \frac{3(2x-3)+16(3x+5)}{2(3x+5)(2x-3)}$$

$$= \frac{54x+71}{2(3x+5)(2x-3)}, \text{ and so}$$

$$y' = \left(\frac{54x+71}{2}\right)(3x+5)^{-1}(2x-3)^{-1}y = \left(\frac{54x+71}{2}\right)(3x+5)^{-1}(2x-3)^{-1}(3x+5)^{1/2}(2x-3)^4$$

$$= \tfrac{1}{2}(2x-3)^3(54x+71)(3x+5)^{-1/2}.$$

45. $y = \dfrac{(2x^2-1)^5}{\sqrt{x+1}}$, so $\ln y = \ln\dfrac{(2x^2-1)^5}{(x+1)^{1/2}} = 5\ln(2x^2-1) - \frac{1}{2}\ln(x+1)$. Thus,

$$\frac{y'}{y} = \frac{20x}{2x^2-1} - \frac{1}{2(x+1)} = \frac{40x(x+1)-(2x^2-1)}{2(2x^2-1)(x+1)} = \frac{38x^2+40x+1}{2(2x^2-1)(x+1)}, \text{ and so}$$

$$y' = \frac{38x^2+40x+1}{2(2x^2-1)(x+1)}\cdot\frac{(2x^2-1)^5}{\sqrt{x+1}} = \frac{(38x^2+40x+1)(2x^2-1)^4}{2(x+1)^{3/2}}.$$

46. $y = \dfrac{\sqrt{4+3x^2}}{\sqrt[3]{x^2+1}}$, so $\ln y = \frac{1}{2}\ln(4+3x^2) - \frac{1}{3}\ln(x^2+1)$. Thus,

$$\frac{y'}{y} = \frac{6x}{2(4+3x^2)} - \frac{2x}{3(x^2+1)} = \frac{9x(x^2+1)-2x(4+3x^2)}{3(4+3x^2)(x^2+1)}, \text{ and so}$$

$$y' = \frac{3x^3+x}{3(4+3x^2)(x^2+1)}\cdot\frac{\sqrt{4+3x^2}}{(x^2+1)^{1/3}} = \frac{x(3x^2+1)}{3(4+3x^2)^{1/2}(x^2+1)^{4/3}}.$$

47. $y = 3^x$, so $\ln y = x\ln 3$, $\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \ln 3$, and $\dfrac{dy}{dx} = y\ln 3 = 3^x\ln 3$.

48. $y = x^{x+2}$, so $\ln y = \ln x^{x+2} = (x+2)\ln x$, So $\dfrac{y'}{y} = \ln x + (x+2)\left(\dfrac{1}{x}\right) = \dfrac{x\ln x + x + 2}{x}$, and

$$y' = \frac{(x\ln x + x + 2)x^{x+2}}{x}.$$

49. $y = (x^2+1)^x$, so $\ln y = \ln(x^2+1)^x = x\ln(x^2+1)$,

$$\frac{y'}{y} = \ln(x^2+1) + x\left(\frac{2x}{x^2+1}\right) = \frac{(x^2+1)\ln(x^2+1)+2x^2}{x^2+1}, \text{ and}$$

$$y' = \frac{\left[(x^2+1)\ln(x^2+1)+2x^2\right](x^2+1)^x}{x^2+1} = (x^2+1)^{x-1}\left[(x^2+1)\ln(x^2+1)+2x^2\right].$$

50. $y = x^{\ln x}$, so $\ln y = \ln(x^{\ln x}) = (\ln x)^2$. Thus, $\dfrac{y'}{y} = 2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$ and so

$$y' = \frac{2\ln x}{x}\cdot x^{\ln x} = 2(\ln x)x^{\ln x - 1}.$$

51. $\dfrac{d}{dx}(\ln y - x\ln x) = \dfrac{d}{dx}(-1)$, so $\dfrac{d}{dx}\ln y - \dfrac{d}{dx}(x\ln x) = 0$, $\dfrac{y'}{y} = \left[\ln x + x\left(\dfrac{1}{x}\right)\right] = \ln x + 1$, and $y' = (\ln x + 1)y$.

52. $\frac{d}{dx}\left(\ln xy - y^2\right) = \frac{d}{dx}(5)$, so $\frac{d}{dx}\left(\ln x + \ln y - y^2\right) = 0$, $\frac{1}{x} + \frac{y'}{y} - 2yy' = 0$, $y + xy' - 2xy^2y' = 0$, $x\left(1 - 2y^2\right)y' = -y$, and so $y' = \frac{y}{x\left(2y^2 - 1\right)}$.

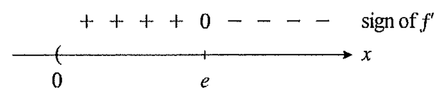
53. $y = x \ln x$. The slope of the tangent line at any point is $y' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$. In particular, the slope of the tangent line at $(1, 0)$ is $m = \ln 1 + 1 = 1$. Thus, an equation of the tangent line is $y - 0 = 1(x - 1)$, or $y = x - 1$.

54. $y = \ln x^2 = 2\ln x$ and $y' = 2/x$, and this gives the slope of the tangent line at any point (x, y) on the graph of $y = \ln x^2$. In particular, the slope of the tangent line at $(2, \ln 4)$ is $m = \frac{2}{2} = 1$. Therefore, an equation is $y - \ln 4 = 1(x - 2)$, or $y = x + \ln 4 - 2$.

55. $f(x) = \ln x^2 = 2\ln x$ and so $f'(x) = 2/x$. Because $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$, we see that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

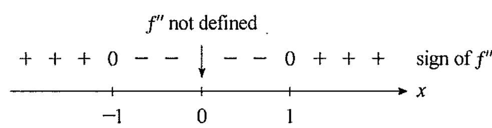
56. $f(x) = \frac{\ln x}{x}$, so $f'(x) = \frac{x\frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$. Observe that $f'(x) = 0$ if $1 - \ln x = 0$, or $x = e$.

The sign diagram of f' on $(0, \infty)$ shows that f is increasing on $(0, e)$ and decreasing on (e, ∞).



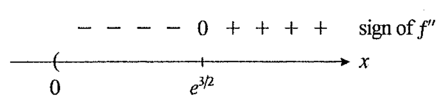
57. $f(x) = x^2 + \ln x^2$, so $f'(x) = 2x + \frac{2x}{x^2} = 2x + \frac{2}{x}$ and $f''(x) = 2 - \frac{2}{x^2}$. To find the intervals of concavity for f, we first set $f''(x) = 0$, giving $2 - \frac{2}{x^2} = 0$, $2 = \frac{2}{x^2}$, $2x^2 = 2$, $x^2 = 1$, and so $x = \pm 1$.

From the sign diagram for f'', we see that f is concave upward on $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 0)$ and $(0, 1)$.

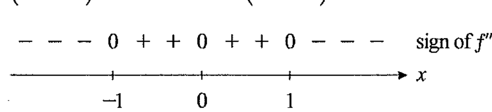


58. $f(x) = \frac{\ln x}{x}$. From Exercise 56, we have $f'(x) = \frac{1 - \ln x}{x^2}$, and so

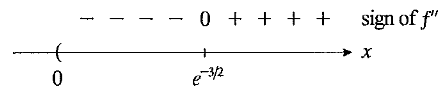
$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4} = \frac{(2\ln x - 3)}{x^3}$. Observe that $f''x) = 0$ implies $2\ln x - 3 = 0$, $\ln x = \frac{3}{2}$, and so $x = e^{3/2}$. From the sign diagram of f'', we see that the graph of f is concave downward on $\left(0, e^{3/2}\right)$ and concave upward on $\left(e^{3/2}, \infty\right)$.



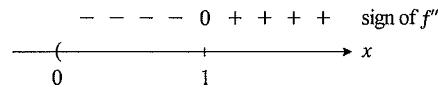
59. $f(x) = \ln\left(x^2 + 1\right)$, so $f'(x) = \frac{2x}{x^2 + 1}$ and $f''(x) = \frac{\left(x^2 + 1\right)(2) - (2x)(2x)}{\left(x^2 + 1\right)^2} = -\frac{2\left(x^2 - 1\right)}{\left(x^2 + 1\right)^2}$. Setting $f''(x) = 0$ gives $x = \pm 1$ as candidates for inflection points of f. From the sign diagram of f'', we see that $(-1, \ln 2)$ and $(1, \ln 2)$ are inflection points of f.



60. $f(x) = x^2 \ln x$, so $f'(x) = 2x \ln x + x^2 \left(\frac{1}{x}\right) = 2x \ln x + x$ and $f''(x) = 2 \ln x + 2x \left(\frac{1}{x}\right) + 1 = 2 \ln x + 3 = 0$ implies that $\ln x = -\frac{3}{2}$, so $x = e^{-3/2}$. From the sign diagram of f'', we see that $\left(e^{-3/2}, -\frac{3}{2}e^{-3}\right)$ is an inflection point of f.



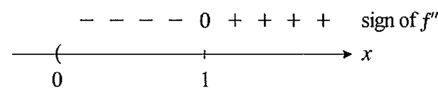
61. $f(x) = x^2 + 2 \ln x$, so $f'(x) = 2x + \dfrac{2}{x}$ and $f''(x) = 2 - \dfrac{2}{x^2} = 0$ implies $2 - \dfrac{2}{x^2} = 0$, $x^2 = 1$, and so $x = \pm 1$. We reject the negative root because the domain of f is $(0, \infty)$. The sign diagram of f'' shows that $(1, 1)$ is an inflection point of the graph of f. $f'(1) = 4$. So, an equation of the required tangent line is $y - 1 = 4(x - 1)$ or $y = 4x - 3$.



62. $f(x) = e^{x/2} \ln x$, so $f'(x) = e^{x/2} \left(\dfrac{1}{x}\right) + \frac{1}{2} e^{x/2} \ln x = \left(\dfrac{1}{x} + \dfrac{\ln x}{2}\right) e^{x/2}$ and

$$f''(x) = \left(-\frac{1}{x^2} + \frac{1}{2x}\right) e^{x/2} + \left(\frac{1}{x} + \frac{\ln x}{2}\right) e^{x/2} \left(\frac{1}{2}\right) = \left(-\frac{1}{x^2} + \frac{1}{x} + \frac{1}{4} \ln x\right) e^{x/2}.$$

Thus, $f''(1) = 0$. The sign diagram of f'' shows that $(1, 0)$ is an inflection point. $f(1) = 0$ and $f'(1) = e^{1/2}$, so an equation of the required tangent line is $y - 0 = \sqrt{e}(x - 1)$ or $y = \sqrt{e}x - \sqrt{e}$.



63. $f(x) = x - \ln x$, so $f'(x) = 1 - \dfrac{1}{x} = \dfrac{x - 1}{x} = 0$ if $x = 1$, a critical point of f. From the table, we see that f has an absolute minimum at $(1, 1)$ and an absolute maximum at $(3, 3 - \ln 3)$.

x	$\frac{1}{2}$	1	3
$f(x)$	$\frac{1}{2} + \ln 2$	1	$3 - \ln 3$

64. $g(x) = \dfrac{x}{\ln x}$, so $g'(x) = \dfrac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2}$. Observe that $g'(x) = 0$ if $x = e$, a critical point of g. From the table, we see that f has an absolute minimum at (e, e) and an absolute maximum at $(5, 3.1067)$.

x	2	e	5
$f(x)$	2.885	e	3.1067

65. $\ln(xy) = x + y$, so $\ln x + \ln y = x + y$. Differentiating with respect to x, we obtain $\dfrac{1}{x} + \dfrac{1}{y} y' = 1 + y'$, $y'\left(\dfrac{1}{y} - 1\right) = 1 - 1/x$, and $y'\left(\dfrac{1 - y}{y}\right) = \dfrac{x - 1}{x}$. Thus, $y' = \dfrac{y(x - 1)}{x(1 - y)}$.

66. $\ln x + e^{-y/x} = 10$. Differentiating with respect to x, we obtain $\dfrac{1}{x} + e^{-y/x} \dfrac{d}{dx}\left(-\dfrac{y}{x}\right) = 0$, so $\dfrac{1}{x} + e^{-y/x}\left(\dfrac{-xy' + y \cdot 1}{x^2}\right) = 0$, $\dfrac{xy' - y}{x^2} \cdot e^{-y/x} = \dfrac{1}{x}$, $xy' - y = xe^{y/x}$, $xy' = y + xe^{y/x}$, and finally $\dfrac{dy}{dx} = \dfrac{y + xe^{y/x}}{x}$.

67. $\ln x + xy = 5$. Differentiating with respect to x, we obtain $\frac{1}{x} + y + xy' = 0$. Differentiating again, we have $-\frac{1}{x^2} + y' + y' + xy'' = 0$, so $xy'' = \frac{1}{x^2} - 2y' = \frac{1 - 2x^2 y'}{x^2}$. But $y' = \frac{-\frac{1}{x} - y}{x} = \frac{1 - xy}{x^2}$, so $y'' = \frac{1}{x^3}\left[1 + 2x^2\left(\frac{1 + xy}{x^2}\right)\right] = \frac{3 + 2xy}{x^3}$.

68. $\ln y + y = x$. Differentiating with respect to x, we obtain $\frac{y'}{y} + y' = 1$, $y'\left(\frac{1}{y} + 1\right) = 1$, $y'\left(\frac{1 + y}{y}\right) = 1$, and $y' = \frac{y}{1 + y}$. Differentiating again, we have $y'' = \frac{(1 + y)\, y' - y\,(y')}{(1 + y)^2} = \frac{y'}{(1 + y)^2} = \frac{y}{1 + y} \cdot \frac{1}{(1 + y)^2} = \frac{y}{(1 + y)^3}$.

69. $\ln y + xy = 1$. Differentiating with respect to x, we obtain $\frac{y'}{y} + y + xy' = 0$. Substituting $x = 1$ and $y = 1$ gives $y' + 1 + y' = 0$, so $y' = -\frac{1}{2}$.

70. $\ln x + xe^y = 1$. Differentiating with respect to x, we obtain $\frac{1}{x} + e^y + xe^y y' = 0$. Substituting $x = 1$ and $y = 0$ gives $1 + 1 + y' = 0$, so $y' = -2$. Thus, an equation of the tangent line at $(1, 0)$ is $y - 0 = -2\,(x - 1)$ or $y = -2x + 2$.

71. $f(x) = 7.2956 \ln\left(0.0645012x^{0.95} + 1\right)$, so

$$f'(x) = 7.2956 \cdot \frac{\frac{d}{dx}\left(0.0645012x^{0.95} + 1\right)}{0.0645012x^{0.95} + 1} = \frac{7.2956\,(0.0645012)\left(0.95x^{-0.05}\right)}{0.0645012x^{0.95} + 1} = \frac{0.4470462}{x^{0.05}\left(0.0645012x^{0.95} + 1\right)}.$$

Thus, $f'(100) = 0.05799$, or approximately 0.0580%/kg, and $f'(500) = 0.01330$, or approximately 0.0133%/kg.

72. $\ln W = \ln 2.4 + 1.84h$. Differentiating this equation implicitly with respect to h yields $\frac{W'}{W} = 1.84$, or $W' = 1.84W$. Therefore, $\Delta W \approx dW = W'\,dh = 1.84W\,dh$. When $h = 1$, $\ln W = \ln 2.4 + 1.84\,(1) \approx 2.71547$, so $W \approx 15.112$. Thus, with $dh = \Delta h = 0.1$, we have $\Delta W = (1.84)\,(15.112)\,(0.1) \approx 2.78061$, and so the weight of the child increases by approximately 2.78 kg.

73. a. $W'(t) = \frac{d}{dt}(49.9 + 17.1 \ln t) = \frac{17.1}{t} > 0$ if $t > 0$, so $W'(t) > 0$ on $[1, 6]$ and W is increasing on $(1, 6)$.

b. $W''(t) = \frac{d}{dt}\left(\frac{17.1}{t}\right) = -\frac{17.1}{t^2} < 0$ on $(1, 6)$, so W is concave downward on $(1, 6)$.

74. a. The amount spent in 2012 was $f(0) = 3.7$ (billion dollars). The amount spent in 2014 was $f(2) = 3.7 + 0.84 \ln(2 + 1)$, or approximately \$4.6 billion.

b. $f'(t) = \frac{0.84}{t + 1}$, so the amount spent annually was growing at the rate of $f'(2) = \frac{0.84}{2 + 1} = 0.28$, or approximately \$0.28 billion/year, in 2014.

75. a. $\ln V = \ln\left(C\left(1-\frac{2}{N}\right)^n\right) = \ln C + n\ln\left(1-\frac{2}{N}\right)$, so $\frac{d}{dn}\ln V = \frac{d}{dn}(\ln C) + \frac{d}{dn}\left[n\ln\left(1-\frac{2}{N}\right)\right]$, $\frac{V'}{V} = \ln\left(1-\frac{2}{N}\right)$, and $V' = V\ln\left(1-\frac{2}{N}\right) = C\left(1-\frac{2}{N}\right)^n\ln\left(1-\frac{2}{N}\right)$.

b. The relative rate of change of $V(n)$ is given by $\frac{V'(n)}{V(n)} = \frac{C\left(1-\frac{2}{N}\right)^n\ln\left(1-\frac{2}{N}\right)}{C\left(1-\frac{2}{N}\right)^n} = \ln\left(1-\frac{2}{N}\right)$.

76. a. $V(2) = 60{,}000\left(1-\frac{2}{10}\right)^2 = 38{,}400$, or \$38,400.

b. $\frac{V'(2)}{V(2)} = \ln\left(1-\frac{2}{10}\right) \approx -0.223$, or approximately -22.3%.

77. $\ln P(t) = \ln\left(\frac{40+80e^{0.06t}}{20+e^{0.06t}}\right) = \ln\left(40+80e^{0.06t}\right) - \ln\left(20+e^{0.06t}\right)$, so

$\frac{P'(t)}{P(t)} = \frac{d}{dt}P(t) = \frac{80\left(0.06e^{0.06t}\right)}{40+80e^{0.06t}} - \frac{0.06e^{0.06t}}{20+e^{0.06t}}$ and $\left.\frac{P'(t)}{P(t)}\right|_{t=60} = \frac{80\left(0.06e^{3.6}\right)}{40+80e^{3.6}} - \frac{0.06e^{3.6}}{20+e^{3.6}} \approx 0.0204$.

Therefore, the relative rate of growth of the population five years after the establishment of the biotech research center is approximately 2.04% per month.

78. a. The revenue function is $R(x) = px = (200 - 0.01x\ln x)x = 200x - 0.01x^2\ln x$ and the marginal revenue function is $R'(x) = 200 - 0.01\left[x^2(1/x) + 2x\ln x\right] = 200 - 0.01x - 0.02x\ln x$.

b. The approximate revenue to be realized from the sale of the 500th yacht is $R'(499) = 200 - 0.01(499) - 0.02(499)\ln 499 \approx 133.01$, or approximately \$13,301.

79. $P(x) = 2\ln(2x+1) + 2x - x^2 - 0.3$. We want to maximize the function P with respect to x. Setting $P'(x) = 0$ gives $P'(x) = \frac{2\cdot 2}{2x+1} + 2 - 2x = 0$, or $4 + (2x+1)(2-2x) = 0$, $2x^2 - x - 3 = 0$, and $(2x-3)(x+1) = 0$. Therefore, $x = -1$ or $x = \frac{3}{2}$. We reject the negative root, so $\frac{3}{2}$ is the only critical number of P. Because $P''(x) = 4\frac{d}{dx}(2x+1)^{-1} - 2 = -\frac{4\cdot 2}{(2x+1)^2} - 2 < 0$ for all $x > 0$, we see that the graph of P is concave downward on $(0,\infty)$, implying that $\frac{3}{2}$ gives an absolute maximum for P with value $P\left(\frac{3}{2}\right) \approx 3.22$. Thus, by employing 150 consultants, Seko makes an estimated annual profit of approximately \$3.22 million.

80. a. $\ln I = \ln I_0 a^x = \ln I_0 + \ln a^x = \ln I_0 + x(\ln a)$. Therefore, $\frac{I'}{I} = \frac{d}{dx}[\ln I_0 + x(\ln a)] = \frac{d}{dx}[x(\ln a)] = \ln a$ and $I' = (\ln a)I = (\ln a)I_0 a^x$.

b. Because $I' = (\ln a)I$, we conclude that I' is proportional to I with $\ln a$ as the constant of proportion.

81. a. $100\frac{d}{dx}[\ln f(x)] = \frac{100f'(x)}{f'(x)}$, and this is precisely the percentage rate of change of f.

b. The percentage rate of growth of the company t years from now is $100\frac{d}{dt}[\ln R(t)] = 100\frac{d}{dt}\left[\ln\left(0.1t^{1.5}e^{0.2t}\right)\right] = 100\frac{d}{dt}(\ln 0.1 + 1.5\ln t + 0.2t) = 100\left(\frac{1.5}{t} + 0.2\right)$.

Thus, the percentage rate of growth 3 years from now is $100\left(\frac{1.5}{3} + 0.2\right) = 70$, or 70%/year.

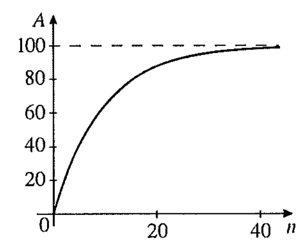
82. a. We find $-\dfrac{\frac{d}{dp}\left[\ln f(p)\right]}{\frac{d}{dp}(\ln p)} = -\dfrac{\frac{f'(p)}{f(p)}}{\frac{1}{p}} = -\dfrac{pf'(p)}{f(p)} = E(p)$, proving the assertion.

b. $$E(p) = -\dfrac{\frac{d}{dp}\left[\ln f(p)\right]}{\frac{d}{dp}(\ln p)} = -\dfrac{\frac{d}{dp}\left[\ln \frac{e^{-0.1p^{1/2}}}{(2p+1)^{1/2}}\right]}{\frac{1}{p}} = -p\frac{d}{dp}\left[\ln e^{-0.1p^{1/2}} - \ln(2p+1)^{1/2}\right]$$
$$= -p\frac{d}{dp}\left[-0.1p^{1/2} - \tfrac{1}{2}\ln(2p+1)\right] = -p\left(-\frac{p^{-1/2}}{20} - \frac{1}{2}\frac{2}{2p+1}\right) = \frac{1}{20}\left(\sqrt{p} + \frac{20p}{2p+1}\right).$$

83. The relative rate of change of P is
$$\frac{P'(t)}{P(t)} = \frac{d}{dt}\left[\ln P(t)\right] = \frac{d}{dt}\ln\left(Le^{-\ln(L/P_0)e^{-ct}}\right) = \frac{d}{dt}\left[\ln L - \ln\left(\frac{L}{P_0}\right)e^{-ct}\right] = 0 - \ln\left(\frac{L}{P_0}\right)\left(-ce^{-ct}\right)$$
$$= c\ln\left(\frac{L}{P_0}\right)e^{-ct}.$$

84. a. If $0 < r < 100$, then $c = 1 - \dfrac{r}{100}$ satisfies $0 < c < 1$. It suffices to show that $A_1(n) = -\left(1 - \frac{r}{100}\right)^n$ is increasing; that is, it suffices to show that $A_2(n) = -A_1(n) = \left(1 - \frac{r}{100}\right)^n$ is decreasing. Let $y = \left(1 - \frac{r}{100}\right)^n$. Then $\ln y = \ln\left(1 - \frac{r}{100}\right)^n = \ln c^n = n\ln c$. Differentiating both sides with respect to n, we find $\frac{y'}{y} = \ln c$, and so $y' = (\ln c)\left(1 - \frac{r}{100}\right)^n$. This is negative because $\ln c < 0$ and $\left(1 - \frac{r}{m}\right)^n > 0$ for $0 < r < 100$. Therefore, A is an increasing function of n on $(0, \infty)$.

b.



c. $\lim\limits_{n\to\infty} A(n) = \lim\limits_{n\to\infty} 100\left[1 - \left(1 - \frac{r}{100}\right)^n\right] = 100.$

85. a. $R = \log\dfrac{10^6 I_0}{I_0} = \log 10^6 = 6.$

b. $I = I_0 10^R$ by definition. Taking the natural logarithm on both sides, we find $\ln I = \ln I_0 10^R = \ln I_0 + \ln 10^R = \ln I_0 + R\ln 10$. Differentiating implicitly with respect to R, we obtain $\dfrac{I'}{I} = \ln 10$. Therefore, $\Delta I \approx dI = \dfrac{dI}{dR}\Delta R = (\ln 10)\, I\, \Delta R$. If $|\Delta R| \le (0.02)(6) = 0.12$ and $I = 1{,}000{,}000 I_0$, (see part (a)), then $|\Delta I| \le (\ln 10)(1{,}000{,}000 I_0)(0.12) \approx 276{,}310.21 I_o$. Thus, the error is at most 276,310 times the standard reference intensity.

86. a. $R(S_0) = k\ln\frac{S_0}{S_0} = k\ln 1 = 0$ because $\ln 1 = 0$.

b. $\dfrac{dR}{dS} = \dfrac{d}{dS}k\ln\dfrac{S}{S_0} = k\dfrac{d}{dS}(\ln S - \ln S_0) = k\dfrac{d}{dS}(\ln S) = \dfrac{k}{S}$, and so $\dfrac{dr}{dS}$ is inversely proportional to S with k as the constant of proportionality. Our result says that if the stimulus is small, then a small change in S is easily felt. But if the stimulus is larger, then a small change in S is not as discernible.

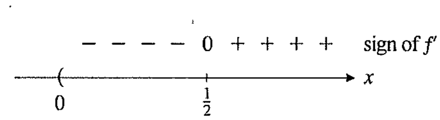
87. $-C\ln y + Dy = A\ln x - Bx + E$. Differentiating implicitly gives $-C\dfrac{y'}{y} + Dy' = \dfrac{Ax'}{x} - Bx'$, $\left(D - \dfrac{C}{y}\right)y' = \left(\dfrac{A}{x} - B\right)x'$, $\left(\dfrac{Dy - C}{y}\right)y' = \left(\dfrac{A - Bx}{x}\right)x'$, and so $y' = \dfrac{(A - Bx)\,yx'}{(Dy - C)\,x}$.

88. We differentiate $Vt = p - k\ln\left(1 - \dfrac{p}{x_0}\right)$ with respect to t, obtaining

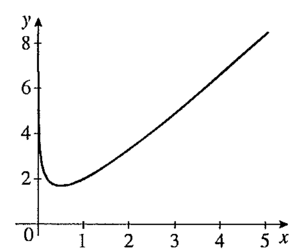
$$V = \frac{dp}{dt} - k\frac{-\dfrac{1}{x_0}\dfrac{dp}{dt}}{1 - \dfrac{p}{x_0}} = \frac{dp}{dt}\left(1 + \frac{k}{x_0 - p}\right) = \frac{dp}{dt}\left(\frac{x_0 - p + k}{x_0 - p}\right).$$ Thus, $\dfrac{dp}{dt} = \dfrac{V(x_0 - p)}{x_0 - p + k}$.

89. $f(x) = 2x - \ln x$. We first gather the following information on f.

1. The domain of f is $(0, \infty)$.
2. There is no y-intercept.
3. $\lim\limits_{x\to\infty}(2x - \ln x) = \infty$.
4. There is no asymptote.
5. $f'(x) = 2 - \dfrac{1}{x} = \dfrac{2x - 1}{x}$. Observe that $f'(x) = 0$ at $x = \frac{1}{2}$, a critical point of f. From the sign diagram of f', we conclude that f is decreasing on $\left(0, \frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2}, \infty\right)$.

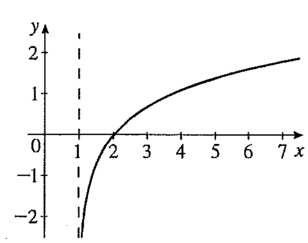


6. The results of part 5 show that $\left(\frac{1}{2}, 1 + \ln 2\right)$ is a relative minimum of f.
7. $f''(x) = \dfrac{1}{x^2}$ and is positive if $x > 0$, so the graph of f is concave upward on $(0, \infty)$.
8. The results of part 7 show that f has no inflection point.



90. $f(x) = \ln(x - 1)$. We first gather the following information on f.

1. The domain of f is obtained by requiring that $x - 1 > 0$. We find the domain to be $(1, \infty)$.
2. Because $x \neq 0$, there is no y-intercept. Next, setting $y = 0$ gives $x - 1 = 1$, so the x-intercept is 2.
3. $\lim\limits_{x\to 1^+}\ln(x - 1) = -\infty$.
4. There is no horizontal asymptote. Because $\lim\limits_{x\to 1^+}\ln(x - 1) = -\infty$, $x = 1$ is a vertical asymptote.
5. $f'(x) = \dfrac{1}{x - 1} > 0$ for $x > 1$, so f has no critical number.
6. The results of part 5 show that f is increasing on $(1, \infty)$.
7. $f''(x) = -\dfrac{1}{(x - 1)^2}$. Because $f''(x) < 0$ for $x > 1$, we see that f is concave downward on $(1, \infty)$.
8. From the results of part 7, we see that f has no inflection point.



91. a. $f(x) = b^x$. Taking the logarithm of each side, we have $\ln f(x) = \ln b^x = x \ln b$, So $\frac{d}{dx}[\ln f(x)] = \frac{d}{dx}(x \ln b)$ and $\frac{f'(x)}{f(x)} = \ln b$. Therefore, $f'(x) = (\ln b) f(x) = (\ln b) b^x$.

b. $f'(x) = \frac{d}{dx}(3^x) = (\ln 3) 3^x$.

92. a. $f(x) = \log_b x$, so $x = b^{f(x)}$. Thus, $\frac{d}{dx}(x) = \frac{d}{dx}\left[b^{f(x)}\right]$, $1 = (\ln b) b^{f(x)} f'(x)$, and therefore $f'(x) = \frac{1}{(\ln b) b^{f(x)}} = \frac{1}{(\ln b) x}$.

b. $f'(x) = \frac{d}{dx}(\log_{10} x) = \frac{1}{(\ln 10) x}$.

93. $f'(x) = \frac{d}{dx}(x^3 2^x) = x^3 \frac{d}{dx}(2^x) + 2^x \frac{d}{dx}(x^3) = (\ln 2) x^3 2^x + 3x^2 2^x = x^2 (x \ln 2 + 3) 2^x$.

94. $g'(x) = \frac{d}{dx}\left(\frac{10^x}{x+1}\right) = \frac{(x+1)(\ln 10) 10^x - 10^x}{(x+1)^2} = \frac{[(x+1)(\ln 10) - 1] 10^x}{(x+1)^2}$.

95. $h'(x) = \frac{d}{dx}(x^2 \log_{10} x) = x^2 \frac{d}{dx} \log_{10} x + (\log_{10} x) \frac{d}{dx}(x^2) = \frac{x^2}{x \ln 10} + 2x \log_{10} x$

$= x\left(\frac{1}{\ln 10} + 2 \log_{10} x\right)$.

96. $f(x) = 3^{x^2} + \log_2(x^2 + 1)$, so

$$f'(x) = \frac{d}{dx}\left[3^{x^2} + \log_2(x^2+1)\right] = (\ln 3) 3^{x^2}(2x) + \frac{2x}{(\ln 2)(x^2+1)}$$

$$= 2x\left[(\ln 3) 3^{x^2} + \frac{1}{(\ln 2)(x^2+1)}\right].$$

97. False. $\ln 5$ is a constant function and $f'(x) = 0$.

98. True. $f(x) = \ln a^x = x \ln a$, so $f'(x) = \frac{d}{dx}(x \cdot \ln a) = \ln a$.

99. If $x \le 0$, then $|x| = -x$. Therefore, $\ln|x| = \ln(-x)$. Writing $f(x) = \ln|x|$, we have $|x| = -x = e^{f(x)}$. Differentiating both sides with respect to x and using the Chain Rule,we have $-1 = e^{f(x)} \cdot f'(x)$, so $f'(x) = -\frac{1}{e^{f(x)}} = -\frac{1}{-x} = \frac{1}{x}$.

100. Let $f(x) = \ln x$. Then by definition, $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{\ln(h+1)}{h}$. But $f'(x) = \frac{d}{dx} \ln x = \frac{1}{x}$, and so with $h = x$, we have $f'(1) = 1 = \lim_{x\to 0} \frac{\ln(x+1)}{x}$.