65. **a.** Taking logarithms of both sides gives $\ln 2^x = \ln e^{kx}$, so $x \ln 2 = kx (\ln e) = kx$. Thus, $x (\ln 2 - k) = 0$ for all x, and this implies that $k = \ln 2$.

b. Proceeding as in part (a), we find that $k = \ln b$.

66. **a.** Let $p = \log_b m$ and $q = \log_b n$, so that $m = b^p$ and $n = b^q$. Then $mn = b^p b^q = b^{p+q}$ and by definition, $p + q = \log_b mn$; that is, $\log_b mn = \log_b m + \log_b n$.

b. $\dfrac{m}{n} = \dfrac{b^p}{b^q} = b^{p-q}$, so by definition, $p - q = \log_b \dfrac{m}{n}$; that is, $\log_b \dfrac{m}{n} = \log_b m - \log_b n$.

67. Let $\log_b m = p$. Then $m = b^p$. Therefore, $m^n = (b^p)^n = b^{np}$, and so $\log_b m^n = \log_b b^{np} = np \log_b b = np$ (since $\log_b b = 1$) $= n \log_b m$.

68. **a.** By definition, $\log_b 1 = 0$ means $1 = b^0 = 1$.

b. By definition, $\log_b b = 1$ means $b = b^1 = b$.

5.3 Compound Interest

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1. **a.** When simple interest is computed, the interest earned is based on the original principal. When compound interest is computed, the interest earned is periodically added to the principal and thereafter earns interest at the same rate.

b. The simple interest formula is $A = P(1 + rt)$ and the compound interest formula is $A = P\left(1 + \dfrac{r}{m}\right)^{mt}$.

2. **a.** The effective rate of interest is the simple interest that would produce the same amount in 1 year as the nominal rate compounded m times per year.

b. $r_{\text{eff}} = \left(1 + \dfrac{r}{m}\right)^m - 1$.

3. $P = A\left(1 + \dfrac{r}{m}\right)^{-mt}$.

4. $A = Pe^{rt}$.

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1. $A = 2500\left(1 + \dfrac{0.04}{2}\right)^{20} = 3714.87$, or \$3714.87.

2. $A = 12{,}000\left(1 + \dfrac{0.05}{4}\right)^{40} = 19{,}723.43$, or \$19,723.43.

3. $A = 150{,}000\left(1 + \dfrac{0.06}{12}\right)^{48} = 190{,}573.37$, or \$190,573.37.

4. $A = 150{,}000\left(1 + \dfrac{0.04}{12}\right)^{1095} = 169{,}123.42$, or \$169,123.42.

5. **a.** Using the formula $r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$ with $r = 0.06$ and $m = 2$, we have $r_{\text{eff}} = \left(1 + \frac{0.06}{2}\right)^2 - 1 = 0.0609$, or 6.09%/yr.

b. Using the formula $r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$ with $r = 0.05$ and $m = 4$, we have $r_{\text{eff}} = \left(1 + \frac{0.05}{4}\right)^4 - 1 = 0.05095$, or 5.095%/yr.

6. **a.** Using the formula $r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$ with $r = 0.045$ and $m = 12$, we have $r_{\text{eff}} = \left(1 + \frac{0.045}{12}\right)^{12} - 1 = 0.04594$, or 4.6%/yr.

b. The effective rate is given by $R = \left(1 + \frac{0.045}{365}\right)^{365} - 1 = 0.04602$, or 4.602%/yr.

7. **a.** The present value is given by $P = 40{,}000\left(1 + \frac{0.05}{2}\right)^{-8} = 32{,}829.86$, or \$32,829.86.

b. The present value is given by $P = 40{,}000\left(1 + \frac{0.05}{4}\right)^{-16} = 32{,}789.85$, or \$32,789.85.

8. **a.** The present value is given by $P = 40{,}000\left(1 + \frac{0.04}{12}\right)^{-48} = 34{,}094.82$, or \$34,094.82.

b. The present value is given by $P = 40{,}000\left(1 + \frac{0.06}{365}\right)^{-(365)(4)} = 31{,}465.74$, or \$31,465.74.

9. $A = 5000e^{0.05(4)} \approx 6107.01$, or \$6107.01.

10. $A = 25000\,(1 + 0.04)^6 \approx 31{,}632.98$, or approximately \$31,632.98. The interest earned is \$6632.98.

11. We use Formula (8) with $A = 10{,}000$, $m = 365$, $r = 0.04$, and $t = 2$. The required deposit is $P = 10{,}000\left(1 + \frac{0.04}{365}\right)^{-365(2)} \approx 9231.20$, or \$9231.20.

12. We use Formula (8) with $A = 15{,}000$, $m = 365$, $r = 0.05$, and $t = 3$. The initial deposit is $P = 15{,}000\left(1 + \frac{0.05}{365}\right)^{-365(3)} \approx 12{,}910.752$, or \$12,910.75.

13. We use Formula (11) with $A = 20{,}000$, $r = 0.06$, and $t = 3$. Jack should deposit $P = 20{,}000e^{-(0.06)(3)} \approx 16{,}705.404$, or \$16,705.40.

14. We use Formula (11) with $A = 12{,}000$, $r = 0.06$, and $t = 2$. Diego's deposit is $P = 12{,}000e^{-(0.06)(2)} \approx 10{,}643.045$, or \$10,643.05.

15. $P = Ae^{-rt} = 59{,}673e^{-(0.06)5} \approx 44{,}206.85$, or approximately \$44,206.85.

16. We use Formula (6) with $A = 7500$, $P = 5000$, $m = 4$, and $t = 3$. Thus, $7500 = 5000\left(1 + \frac{r}{4}\right)^{12}$, so $\left(1 + \frac{r}{4}\right)^{12} = \frac{7500}{5000} = \frac{3}{2}$, $\ln\left(1 + \frac{r}{4}\right)^{12} = \ln 1.5$, $12\ln\left(1 + \frac{r}{4}\right) = \ln 1.5$, $\ln\left(1 + \frac{r}{4}\right) = \frac{\ln 1.5}{12} \approx 0.0337888$, $1 + \frac{r}{4} \approx e^{0.0337888} \approx 1.034366$, $\frac{r}{4} \approx 0.034366$, and $r \approx 0.137464$. The required annual interest rate is 13.75%.

17. We use Formula (6) with $A = 7500$, $P = 5000$, $m = 12$, and $t = 3$. Thus, $7500 = 5000\left(1 + \frac{r}{12}\right)^{36}$, $\left(1 + \frac{r}{12}\right)^{36} = \frac{7500}{5000} = \frac{3}{2}$, $\ln\left(1 + \frac{r}{12}\right)^{36} = \ln 1.5$, $36\ln\left(1 + \frac{r}{12}\right) = \ln 1.5$, $\ln\left(1 + \frac{r}{12}\right) = \frac{\ln 1.5}{36} = 0.0112629$, $1 + \frac{r}{12} = e^{0.0112629} = 1.011327$, $\frac{r}{12} = 0.011327$, and $r = 0.13592$. The annual interest rate is 13.59%.

18. We use Formula (6) with $A = 8000$, $P = 5000$, $m = 2$, and $t = 4$. Thus, $8000 = 5000\left(1 + \frac{r}{2}\right)^{8}$, $\left(1 + \frac{r}{2}\right)^{8} = \frac{8000}{5000} = 1.6$, $\ln\left(1 + \frac{r}{2}\right)^{8} = \ln 1.6$, $8\ln\left(1 + \frac{r}{2}\right) = \ln 1.6$, $\ln\left(1 + \frac{r}{2}\right) = \frac{\ln 1.6}{8} = 0.05875$, $1 + \frac{r}{2} = e^{0.05875} = 1.06051$; $\frac{r}{2} = 0.06051$, and so $r = 0.1210$. The required annual interest rate is 12.1%.

19. We use Formula (6) with $A = 5500$, $P = 5000$, $m = 12$, and $t = \frac{1}{2}$. Thus, $5500 = 5000\left(1 + \frac{r}{12}\right)^{6}$, and so $\left(1 + \frac{r}{12}\right)^{6} = \frac{5500}{5000} = 1.1$. Proceeding as in the previous exercise, we find $r = 0.1921$, so the required annual interest rate is 19.21%.

20. We use Formula (6) with $A = 4000$, $P = 2000$, $m = 1$, and $t = 5$. Thus, $4000 = 2000(1 + r)^5$, $(1 + r)^5 = 2$, $5\ln(1 + r) = \ln 2$, $\ln(1 + r) = \frac{\ln 2}{5} \approx 0.138629$, $1 + r \approx e^{0.138629} \approx 1.148698$, and so $r \approx 0.1487$. The required annual interest rate is 14.87%.

21. We use Formula (6) with $A = 6000$, $P = 2000$, $m = 12$, and $t = 5$. Thus, $6000 = 2000\left(1 + \frac{r}{12}\right)^{60}$. Thus, $\left(1 + \frac{r}{12}\right)^{60} = 3$, $60\ln\left(1 + \frac{r}{12}\right) = \ln 3$, $\ln\left(1 + \frac{r}{12}\right) = \frac{\ln 3}{60}$, $1 + \frac{r}{12} = e^{(\ln 3)/60}$, $\frac{r}{12} = e^{(\ln 3)/60} - 1$, and $r = 12\left(e^{(\ln 3)/60} - 1\right) \approx 0.2217$, so the required interest rate is 22.17% per year.

22. We use Formula (6) with $A = 15000$, $P = 12000$, $m = 12$, and $r = 0.05$. Thus, $15000 = 12000\left(1 + \frac{0.05}{12}\right)^{12t}$, $\left(1 + \frac{0.05}{12}\right)^{12t} = \frac{15{,}000}{12{,}000} = 1.25$, $12t\ln\left(1 + \frac{0.05}{12}\right) = \ln 1.25$, and so $t = \dfrac{\ln 1.25}{12\ln\left(1 + \frac{0.05}{12}\right)} \approx 4.47$. Thus, it will take approximately 4.5 years.

23. We use Formula (6) with $A = 6500$, $P = 5000$, $m = 12$, and $r = 0.06$. Thus, $6500 = 5000\left(1 + \frac{0.06}{12}\right)^{12t}$, $(1.005)^{12t} = \frac{6500}{5000} = 1.3$, $12t\ln(1.005) = \ln 1.3$, and so $t = \dfrac{\ln 1.3}{12\ln 1.005} \approx 4.384$. It will take approximately 4.4 years.

24. We use Formula (6) with $A = 4000$, $P = 2000$, $m = 12$, and $r = 0.06$. Thus, $4000 = 2000\left(1 + \frac{0.06}{12}\right)^{12t}$, $\left(1 + \frac{0.06}{12}\right)^{12t} = 2$, $12t\ln\left(1 + \frac{0.06}{12}\right) = \ln 2$, and so $t = \dfrac{\ln 2}{12\ln\left(1 + \frac{0.06}{12}\right)} \approx 11.58$. It will take approximately 11.6 years.

25. We use Formula (6) with $A = 15000$, $P = 5000$, $m = 365$, and $r = 0.04$. Thus, $15{,}000 = 5000\left(1 + \dfrac{0.04}{365}\right)^{365t}$, from which we find $t = \dfrac{\ln\left(\frac{15{,}000}{5000}\right)}{365\ln\left(1 + \frac{0.04}{365}\right)} \approx 27.47$. Thus, it will take approximately 27.5 years.

26. We use Formula (10) with $A = 6000$, $P = 5000$, and $t = 3$. Thus, $6000 = 5000e^{3r}$, $e^{3r} = \frac{6000}{5000} = 1.2$, $3r = \ln 1.2$, and $r = \frac{1}{3}\ln 1.2 \approx 0.06077$. The annual interest rate is 6.08%.

27. We use Formula (10) with $A = 8000$, $P = 4000$, and $t = 5$. Thus, $8000 = 4000e^{5r}$, $e^{5r} = \frac{8000}{4000} = 2$, $5r = \ln 2$, and $r = \dfrac{\ln 2}{5} \approx 0.13863$. The annual interest rate is 13.86%.

28. We use Formula (10) with $A = 7000$, $P = 6000$, and $r = 0.075$. Thus, $7000 = 6000e^{0.075t}$, $e^{0.075t} = \frac{7000}{6000} = \frac{7}{6}$, $0.075t \ln e = \ln \frac{7}{6}$, and so $t = \dfrac{\ln \frac{7}{6}}{0.075} \approx 2.055$. It will take 2.06 years.

29. We use Formula (10) with $A = 16{,}000$, $P = 8000$, and $r = 0.05$. Thus, $16{,}000 = 8000e^{0.05t}$, and we find that $t = \frac{\ln 2}{0.05} \approx 13.863$. It will take 13.9 years.

30. The Estradas can expect to pay $180{,}000\,(1 + 0.04)^4$, or approximately \$210,575.

31. The utility company will have to increase its generating capacity by a factor of $(1.08)^{10} \approx 2.16$.

32. The investment will be worth $A = 1.5\left(1 + \dfrac{0.025}{2}\right)^{20} \approx 1.9231$, or approximately \$1.9231 million.

33. After 1 year, Maria's investment is worth $(1.2)\,(10{,}000)$ dollars and after 2 years, it is worth $(1.1)\,(1.2)\,(10{,}000)$ dollars.
After 1 year, Laura's investment is worth $(1.1)\,(10{,}000)$ dollars and after 2 years, it is worth $(1.2)\,(1.1)\,(10{,}000)$ dollars.
So after 2 years, both investments are worth the same amount, namely \$13,200.

34. The value of Alan's stock portfolio after 1 year is $(1.2)\,P$ dollars, where P is the original amount invested. Its value after 2 years is $(1.1)\,(1.2)\,P$; after 3 years, it is $(0.9)\,(1.1)\,(1.2)\,P$; and finally after 4 years, it is $(0.8)\,(0.9)\,(1.1)\,(1.2)\,P$ or $0.9504P$ dollars. Thus, the value of Alan's stock portfolio after 4 years is less than the its initial value.

35. Suppose Jack's portfolio is worth $\$P$ initially. After 1 year, it is worth $(0.8)\,P$ dollars, and after 2 years, it is worth $(1.2)\,(0.8)\,P$ or $0.96P$ dollars. This shows that after the second year, Jack's investment has not recouped all of its losses from the first year.

36. Suppose Arabella's stock portfolio is worth $\$P$ initially. Then after 1 year, it is worth $(0.8)\,P$ dollars. Let r denote the annual rate (compounded annually) which the portfolio must earn in the second year in order to regain its original value at the end of the third year. Then $(1 + r)^2\,(0.8)\,P = P$, so $(1 + r)^2 = \frac{1}{0.8}$, $1 + r = \sqrt{\frac{1}{0.8}} \approx 1.1180$, and $r \approx 0.1180$. The required rate is thus approximately 11.8% per year.

37. We use Formula 3 with $P = 15{,}000$, $r = 0.078$, $m = 12$, and $t = 4$, giving the worth of Jodie's account as $A = 15{,}000\left(1 + \frac{0.078}{12}\right)^{(12)(4)} \approx 20{,}471.641$, or approximately \$20,471.64.

38. If the money earns interest at the rate of 6% compounded annually, he receives $A = (1.06)^{21}\,(10{,}000) \approx 33{,}995.64$, or \$33,995.64. If the money earns interest at the rate of 6% compounded quarterly, he receives $A = \left(1 + \frac{0.06}{4}\right)^{4(21)}(10{,}000) \approx 34{,}925.90$, or \$34,925.90. If the money earns interest at the rate of 6% compounded monthly, he receives $A = \left(1 + \frac{0.06}{12}\right)^{12(21)}(10{,}000) \approx 35{,}143.71$, or \$35,143.71.

39. We use Formula 3 with $P = 10{,}000$, $r = 0.0682$, $m = 4$, and $t = \frac{11}{2}$, giving the worth of Chris' account as $A = 10{,}000\left(1 + \frac{0.0682}{4}\right)^{(4)(11/2)} \approx 14{,}505.433$, or approximately \$14,505.43.

40. **a.** The accumulated amount before taxes is $A = 25{,}000\left(1 + \frac{0.06}{1}\right)^{10} \approx 44{,}771.19$. After taxes, it is worth \$33,235.26.

b. The accumulated tax-free amount is $A = 25{,}000\,(1 + 0.0432)^{10} \approx 38{,}160.65$, or \$38,160.65.

41. He can expect the minimum revenue for 2016 to be $240{,}000\,(1.2)\,(1.3)\,(1.25)^3 \approx 731{,}250$, or \$731,250.

42. The projected online sales for 2009 are $141.4\,(1.243)\,(1.14)\,(1.305)\,(1.176)\,(1.105) \approx 339.79$, or approximately \$339.79 billion.

43. We want the value of a 2013 dollar at the beginning of 2009. Denoting this value by x, we have $(1.027)\,(1.015)\,(1.030)\,(1.017)\,x = 1$, so $x \approx 0.916$. Thus, the purchasing power is approximately 92 cents.

44. **a.** If they invest the money at 4.6% compounded quarterly, they should set aside $P = 120{,}000\left(1 + \frac{0.046}{4}\right)^{-28} \approx 87{,}123.7$, or \$87,123.70.

b. If they invest the money at 4.6% compounded continuously, they should set aside $P = 120{,}000e^{-(0.046)(7)} \approx 86{,}963.8$, or \$86, 963.80.

45. He needs $65{,}000e^{0.03(10)} \approx 87{,}740.82$ or approximately \$87,740.82 annually.

46. Bernie originally invested $P = 22{,}289.22\left(1 + \dfrac{0.03}{4}\right)^{-20} \approx 19{,}195.25$, or \$19,195.25.

47. The present value of the \$8000 loan due in 3 years is given by $P = 8000\left(1 + \dfrac{0.08}{2}\right)^{-6} \approx 6322.52$, or \$6322.52.

The present value of the \$15,000 loan due in 6 years is given by $P = 15{,}000\left(1 + \dfrac{0.08}{2}\right)^{-12} \approx 9368.96$, or \$9368.96.

Therefore, the amount the proprietors of the inn will be required to pay at the end of 5 years is given by $A = 15{,}691.48\left(1 + \dfrac{0.08}{2}\right)^{10} \approx 23{,}227.22$, or \$23,227.22.

48. **a.** If inflation over the next 15 years is 3%, then the first year of Eleni's pension will be worth $P_3 = 40{,}000e^{-0.03(15)} = 25{,}505.13$, or \$25,505.13.

b. If inflation over the next 15 years is 4%, then the first year of Eleni's pension will be worth $P_4 = 40{,}000e^{-0.04(15)} = 21{,}952.47$, or \$21,952.47.

c. If inflation over the next 15 years is 6%, then the first year of Eleni's pension will be worth $P_6 = 40{,}000e^{-0.06(15)} = 16{,}262.79$, or \$16,262.79.

49. $P(t) = V(t)\,e^{-rt} = 80{,}000e^{\sqrt{t}/2}e^{-rt} = 80{,}000e^{(\sqrt{t}/2)-0.09t}$. Thus, $P(4) = 80{,}000e^{1-0.09(4)} \approx 151{,}718.47$, or approximately \$151,718.

50. **a.** Using Formula (11) with $A = V(t)$ and $r = 0.08$, we find $P(t) = V(t)e^{-0.08t} = 500{,}000e^{-0.08t+0.5t^{0.4}}$, $0 \le t \le 8$.

b.

t	4	5	6
$P(t)$	\$867,104	\$868,211	\$861,301

From the table, we see that the present value of the mall seems to decrease after a certain period of growth. We see that sometime between $t = 5$ and $t = 6$, the present value of the mall attains its highest market value, at least \$868,211.

51. Suppose \$1 is invested in each investment. The accumulated amount in investment A is $\left(1 + \frac{0.08}{2}\right)^8 \approx 1.36857$ and the accumulated amount in investment B is $e^{0.0775(4)} \approx 1.36343$. Thus, investment A has a higher rate of return.

52. We solve the equation $366{,}000 = 300{,}000(1+r)^6$, or $(1+r)^6 = 1.22$, obtaining $1 + r = (1.22)^{1/6} \approx 1.0337$, and so $r \approx 0.037$, or 3.37%.

53. The effective annual rate of return on his investment is found by solving the equation $(1+r)^2 = \frac{32{,}100}{25{,}250}$. We find $1 + r = \left(\frac{32{,}100}{25{,}250}\right)^{1/2}$, so $1 + r \approx 1.1275$, and $r \approx 0.1275$, or 12.75%.

54. We solve the equation $3.6 = 1.4e^{6r}$, finding $e^{6r} = \frac{3.6}{1.4}$, $6r \ln e \approx \ln \frac{3.6}{1.4}$, $6r \approx 0.944462$, and so $r \approx 0.1574$, or approximately 15.7%.

55. $r_{\text{eff}} = \lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^m - 1 = e^r - 1.$

56. **a.** $r_{\text{eff}} = \left(1 + \frac{0.1}{4}\right)^4 - 1 \approx 0.1038$, or 10.38%.

b. $r_{\text{eff}} = \left(1 + \frac{0.1}{12}\right)^{12} - 1 \approx 0.1047$, or 10.47%.

c. $r_{\text{eff}} = e^{0.1} - 1 \approx 0.1052$, or 10.52%.

57. The effective rate of interest at Bank A is given by $R = \left(1 + \frac{0.07}{4}\right)^4 - 1 = 0.07186$, or 7.186%. The effective rate at Bank B is given by $R = e^r - 1 = e^{0.07125} - 1 = 0.07385$, or 7.385%. We conclude that Bank B has the higher effective rate of interest.

58. $\left(1 + \frac{r}{12}\right)^{12} - 1 = r_{\text{eff}}$, $\left(1 + \frac{r}{12}\right)^{12} = 1.06$, $1 + \frac{r}{12} = (1.06)^{1/12}$, $\frac{r}{12} = (1.06)^{1/12} - 1$, and so $r = 12\left[(1.06)^{1/12} - 1\right] = 0.05841$, or 5.84%.

59. By definition, $A = P(1 + r_{\text{eff}})^t$, so $(1 + r_{\text{eff}})^t = \frac{A}{P}$, $1 + r_{\text{eff}} = \left(\frac{A}{P}\right)^{1/t}$, and $r_{\text{eff}} = \left(\frac{A}{P}\right)^{1/t} - 1$.

60. According to the result of Exercise 59, $r_{\text{eff}} = \left(\frac{A}{P}\right)^{1/t} - 1$. Here $P = 40{,}000$, $A = 60{,}000$, and $t = 5$, so the required effective rate is $r_{\text{eff}} = \left(\frac{60{,}000}{40{,}000}\right)^{1/5} - 1 \approx 0.0845$, or 8.45%.

61. Using the formula $r_{\text{eff}} = \left(\frac{A}{P}\right)^{1/t} - 1$ with $A = 5070.42$, $P = 5000$, and $t = \frac{245}{365}$, we have $r_{\text{eff}} = \left(\frac{5070.42}{5000}\right)^{1/(245/365)} - 1 = \left(\frac{5070.42}{5000}\right)^{(365/245)} - 1 \approx 0.0211$, or 2.11%.