5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5.1 Exponential Functions

Concept Questions
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1. $f(x) = b^x$ with $b > 0$ and $b \neq 1$.

2. **a.** $y = b^x, b > 0, b \neq 1$ has domain $(-\infty, \infty)$ and range $(0, \infty)$.

   b. The y-intercept is 1.

   c. The function is continuous on $(-\infty, \infty)$.

   d. The function is increasing on $(-\infty, \infty)$ if $b > 1$ and decreasing on $(-\infty, \infty)$ if $b < 1$.

Exercises
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1. **a.** $4^{-3} \times 4^5 = 4^{-3+5} = 4^2 = 16.$ **b.** $3^{-3} \times 3^6 = 3^{6-3} = 3^3 = 27.$

2. **a.** $\left(2^{-1}\right)^3 = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$ **b.** $\left(3^{-2}\right)^3 = 3^{-6} = \frac{1}{3^6} = \frac{1}{729}.$

3. **a.** $9\,(9)^{-1/2} = \frac{9}{9^{1/2}} = \frac{9}{3} = 3.$ **b.** $5\,(5)^{-1/2} = 5^{1/2} = \sqrt{5}.$

4. **a.** $\left[\left(-\frac{1}{2}\right)^3\right]^{-2} = \left(-\frac{1}{2}\right)^{-6} = \frac{(-1)^{-6}}{2^{-6}} = 2^6 = 64.$ **b.** $\left[\left(-\frac{1}{3}\right)^2\right]^{-3} = \left(-\frac{1}{3}\right)^{-6} = \frac{(-1)^{-6}}{3^{-6}} = 3^6 = 729.$

5. **a.** $\frac{(-3)^4(-3)^5}{(-3)^8} = (-3)^{4+5-8} = (-3)^1 = -3.$ **b.** $\frac{\left(2^{-4}\right)\left(2^6\right)}{2^{-1}} = 2^{-4+6+1} = 2^3 = 8.$

6. **a.** $3^{1/4} \times 9^{-5/8} = 3^{1/4}\left(3^2\right)^{-5/8} = 3^{1/4} \times 3^{-5/4}$
   $= 3^{(1/4)-(5/4)} = 3^{-1} = \frac{1}{3}.$

   b. $2^{3/4} \times 4^{-3/2} = 2^{3/4}\left(2^2\right)^{-3/2} = 2^{3/4} \times 2^{-3}$
   $= 2^{(3/4)-3} = 2^{-9/4} = \frac{1}{2^{9/4}}.$

7. **a.** $\frac{5^{3.3} \cdot 5^{-1.6}}{5^{-0.3}} = \frac{5^{3.3-1.6}}{5^{-0.3}} = 5^{1.7+(0.3)} = 5^2 = 25.$ **b.** $\frac{4^{2.7} \cdot 4^{-1.3}}{4^{-0.4}} = 4^{2.7-1.3+0.4} = 4^{1.8} \approx 12.126.$

8. **a.** $\left(\frac{1}{16}\right)^{-1/4}\left(\frac{27}{64}\right)^{-1/3} = (16)^{1/4}\left(\frac{64}{27}\right)^{1/3} = 2\left(\frac{4}{3}\right) = \frac{8}{3}.$ **b.** $\frac{8}{27}^{-1/3}\left(\frac{81}{256}\right)^{-1/4} = \left(\frac{27}{8}\right)^{1/3}\left(\frac{256}{81}\right)^{1/4} = \frac{3}{2} \cdot \frac{4}{3} = 2.$

9. **a.** $\left(64x^9\right)^{1/3} = 64^{1/3}\left(x^{9/3}\right) = 4x^3.$

   b. $\left(25x^3y^4\right)^{1/2} = \left(25^{1/2}\right)\left(x^{3/2}\right)\left(y^{4/2}\right) = 5x^{3/2}y^2$
   $= 5xy^2\sqrt{x}.$

10. **a.** $\left(2x^3\right)\left(-4x^{-2}\right) = -8x^{3-2} = -8x.$ **b.** $\left(4x^{-2}\right)\left(-3x^5\right) = -12x^{-2+5} = -12x^3.$

11. a. $\dfrac{6a^{-4}}{3a^{-3}} = 2a^{-4+3} = 2a^{-1} = \dfrac{2}{a}$.

b. $\dfrac{4b^{-4}}{12b^{-6}} = \frac{1}{3}b^{-4+6} = \frac{1}{3}b^2$.

12. a. $y^{-3/2}y^{5/3} = y^{(-3/2)+(5/3)} = y^{1/6}$.

b. $x^{-3/5}x^{8/3} = x^{(-3/5)+(8/3)} = x^{31/15}$.

13. a. $\left(2x^3y^2\right)^3 = 2^3 \times x^{3(3)} \times y^{2(3)} = 8x^9y^6$.

b. $\left(4x^2y^2z^3\right)^2 = 4^2 \cdot x^{2(2)} \cdot y^{2(2)} \cdot z^{3(2)} = 16x^4y^4z^6$.

14. a. $\left(x^{r/s}\right)^{s/r} = x^{(r/s)(s/r)} = x$.

b. $\left(x^{-b/a}\right)^{-a/b} = x^{(-b/a)(-a/b)} = x$.

15. a. $\dfrac{5^0}{\left(2^{-3}x^{-3}y^2\right)^2} = \dfrac{1}{2^{-3(2)}x^{-3(2)}y^{2(2)}} = \dfrac{2^6x^6}{y^4} = \dfrac{64x^6}{y^4}$.

b. $\dfrac{(x+y)(x-y)}{(x-y)^0} = (x+y)(x-y)$.

16. a. $\dfrac{\left(a^m \cdot a^{-n}\right)^{-2}}{\left(a^{m+n}\right)^2} = \dfrac{a^{-2m} \cdot a^{2n}}{a^{2(m+n)}} = a^{-2m+2n-2(m+n)} = \dfrac{1}{a^{4m}}$.

b. $\left(\dfrac{x^{2n-2}y^{2n}}{x^{5n+1}y^{-n}}\right)^{1/3} = \left(\dfrac{y^{3n}}{x^{3n+3}}\right)^{1/3} = \dfrac{y^n}{x^{n+1}}$.

17. $6^{2x} = 6^6$ if and only if $2x = 6$ or $x = 3$.

18. $5^{-x} = 5^3$ if and only if $-x = 3$ or $x = -3$.

19. $3^{3x-4} = 3^5$ if and only if $3x - 4 = 5$, $3x = 9$, or $x = 3$.

20. $10^{2x-1} = 10^{x+3}$ if and only if $2x - 1 = x + 3$, or $x = 4$.

21. $(2.1)^{x+2} = (2.1)^5$ if and only if $x + 2 = 5$, or $x = 3$.

22. $(-1.3)^{x-2} = (-1.3)^{2x+1}$ if and only if $x - 2 = 2x + 1$, or $x = -3$.

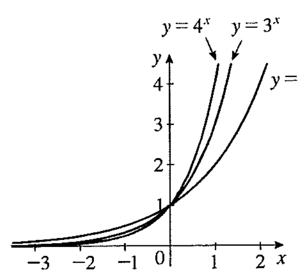
23. $8^x = \left(\frac{1}{32}\right)^{x-2}$, $\left(2^3\right)^x = (32)^{2-x} = \left(2^5\right)^{2-x}$, so $2^{3x} = 2^{5(2-x)}$, $3x = 10 - 5x$, $8x = 10$, or $x = \frac{5}{4}$.

24. $3^{x-x^2} = \dfrac{1}{9^x} = \left(3^2\right)^{-x} = 3^{-2x}$. This is true if and only if $x - x^2 = -2x$, $x^2 - 3x = x(x-3) = 0$, so $x = 0$ or 3.

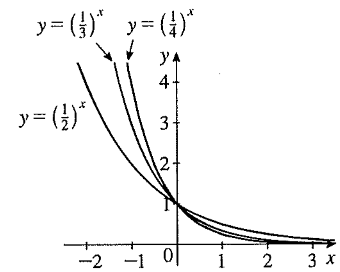
25. Let $y = 3^x$. Then the given equation is equivalent to $y^2 - 12y + 27 = 0$, or $(y-9)(y-3) = 0$, giving $y = 3$ or 9. So $3^x = 3$ or $3^x = 9$, and therefore, $x = 1$ or $x = 2$.

26. $2^{2x} - 4 \cdot 2^x + 4 = 0$, $(2^x)^2 - 4(2^x) + 4 = 0$. Let $y = 2^x$, so $y^2 - 4y + 4 = (y-2)^2 = 0$, or $y = 2$. Thus, $2^x = 2$, or $x = 1$.

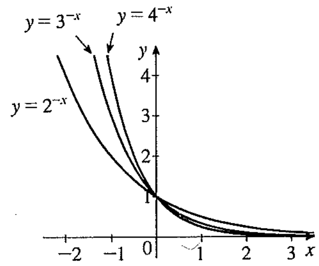
27. $y = 2^x$, $y = 3^x$, and $y = 4^x$.



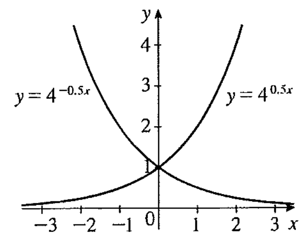
28. $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, and $y = \left(\frac{1}{4}\right)^x$.



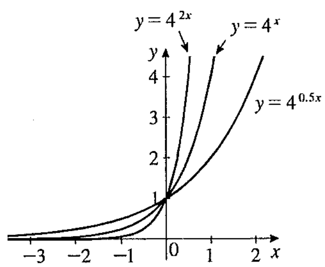
29. $y = 2^{-x}$, $y = 3^{-x}$, and $y = 4^{-x}$.



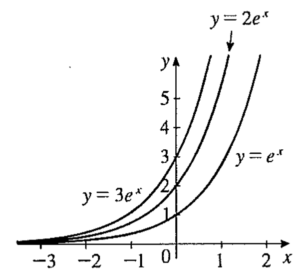
30. $y = 4^{0.5x}$ and $y = 4^{-0.5x}$.



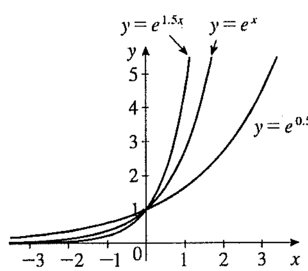
31. $y = 4^{0.5x}$, $y = 4^{x}$, and $y = 4^{2x}$.



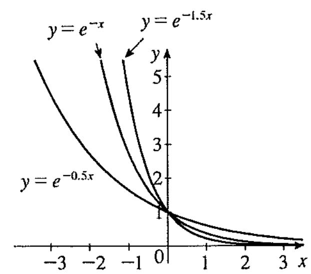
32. $y = e^{x}$, $y = 2e^{x}$, and $y = 3e^{x}$.



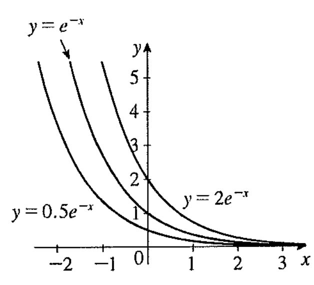
33. $y = e^{0.5x}$, $y = e^{x}$, $y = e^{1.5x}$.



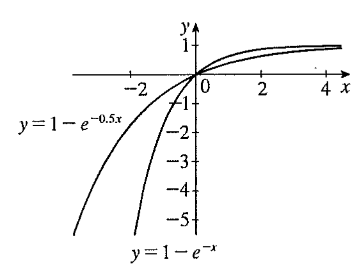
34. $y = e^{-0.5x}$, $y = e^{-x}$, and $y = e^{-1.5x}$.



35. $y = 0.5e^{-x}$, $y = e^{-x}$, and $y = 2e^{-x}$.



36. $y = 1 - e^{-x}$ and $y = 1 - e^{-0.5x}$.



37. Because $f(0) = A = 100$ and $f(1) = 120$, we have $100e^{k} = 120$, and so $e^{k} = \frac{12}{10} = \frac{6}{5}$. Therefore, $f(x) = 100e^{kx} = 100\left(e^{k}\right) = 100\left(\frac{6}{5}\right)^{x}$.

38. Because $f(1) = 5$, $Ae^{-k} = 5$ and $e^{-k} = \dfrac{5}{A}$. Next, $f(2) = 7$ and so $2Ae^{-2k} = 2A\left(e^{-k}\right)^2 = 2A\left(\dfrac{5}{A}\right)^2 = 7$,

$2A\left(\dfrac{25}{A^2}\right) = 7$, $\dfrac{50}{A} = 7$, and so $A = \frac{50}{7}$. Finally, $f(3) = 3Ae^{-3k} = 3A\left(e^{-k}\right)^3 = 3\left(\dfrac{50}{7}\right)\left(\dfrac{5}{\frac{50}{7}}\right)^3 = 7.35$.

39. $f(0) = 20$ implies that $\dfrac{1000}{1+B} = 20$, so $1000 = 20 + 20B$, or $B = \dfrac{980}{20} = 49$. Therefore, $f(t) = \dfrac{1000}{1+49e^{-kt}}$. Next, $f(2) = 30$, so $\dfrac{1000}{1+49e^{-2t}} = 30$. We have $1 + 49e^{-2k} = \dfrac{1000}{30} = \dfrac{100}{3}$, $49e^{-2k} = \dfrac{100}{3} - 1 = \dfrac{97}{3}$, $e^{-2k} = \dfrac{97}{147}$, and finally $e^{-k} = \left(\dfrac{97}{147}\right)^{1/2}$. Therefore, $f(t) = \dfrac{1000}{1+49\left(\frac{97}{147}\right)^{t/2}}$, so

$f(5) = \dfrac{1000}{1+49\left(\frac{97}{147}\right)^{5/2}} \approx 54.6$.

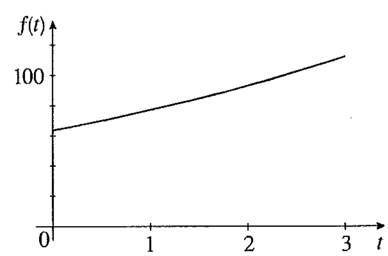
40. a. The average number of viewers in the 2011 season was $f(1) = 32.744e^{-0.252(1)} \approx 25.450$, or approximately 25.450 million.

b. The average number of viewers in the 2014 season was $f(4) = 32.744e^{-0.252(4)} = 11.950$, or approximately 11.950 million.

41. a. $f(t) = 64e^{0.188t}$.

t	0	1	2	3
$f(t)$	64	77.2	93.2	112.5

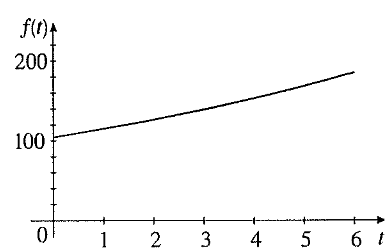
b.



42. a. $f(t) = 105e^{0.095t}$.

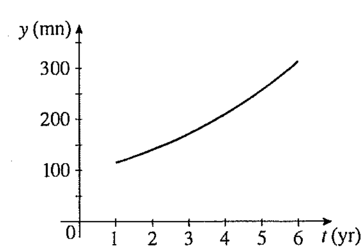
t	1	2	3	4	5	6
$f(t)$	115.5	127.0	139.6	153.5	168.8	185.7

b.



43. a. The number of internet users in 2005 was $f(1) = 115.423$, or 115,423,000. In 2006, it was $f(2) = 94.5e^{0.2(2)} \approx 140.977$, or 140,977,000. The number of internet users in 2010 was $f(6) = 94.5e^{1.2} \approx 313.751$, or 313,751,000.

b.



44. $N(t) = \dfrac{385.474}{1 + 2.521e^{-0.214t}}$. The number of cellphone subscribers in 2000 was $N(0) = \dfrac{385.474}{1 + 2.521} \approx 109.48$, or approximately 109.5 million. The number in 2012 was $N(12) = \dfrac{385.4}{1 + 2.521e^{-0.214(12)}} \approx 322.96$, or approximately 323.0 million.

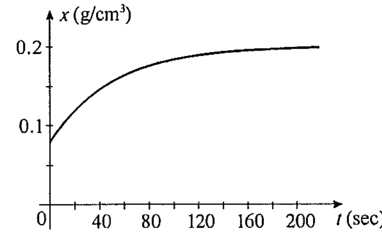
45. $N(t) = \dfrac{35.5}{1 + 6.89e^{-0.8674t}}$, so $N(6) = \dfrac{35.5}{1 + 6.89e^{-0.8674(6)}} \approx 34.2056$, or 34.21 million.

46. **a.** The initial concentration is given by
$x(0) = 0.08 + 0.12\left(1 - e^{-0.02 \cdot 0}\right) = 0.08$, or 0.08 g/cm^3.

b. The concentration after 20 seconds is given by
$x(20) = 0.08 + 0.12\left(1 - e^{-0.02 \cdot 20}\right) = 0.11956$, or 0.1196 g/cm^3.

c. The concentration in the long run is given by
$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \left[0.08 + 0.12\left(1 - e^{-0.02t}\right)\right] = 0.2$, or 0.2 g/cm^3.

d.



47. **a.** The initial concentration is given by $C(0) = 0.3(0) - 18\left(1 - e^{-0/60}\right)$, or 0 g/cm^3.

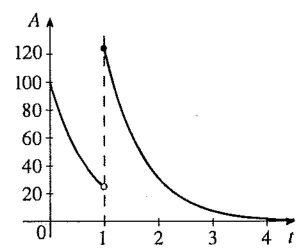
b. The concentration after 10 seconds is given by $C(10) = 0.3(10) - 18\left(1 - e^{-10/60}\right) = 0.23667$, or 0.2367 g/cm^3.

c. The concentration after 30 seconds is given by $C(30) = 18e^{-30/60} - 12e^{-(30-20)/60} = 0.75977$, or 0.7598 g/cm^3.

d. The concentration of the drug in the long run is given by $\lim_{t \to \infty} C(t) = \lim_{t \to \infty} \left(18e^{-t/60} - 12e^{-(t-20)/60}\right) = 0$.

48. **a.** The amount of drug in Jane's body immediately after the second dose is $A(1) = 100\left(1 + e^{1.4}\right)e^{-1.4(1)} = 100\left(e^{-1.4} + 1\right)$, or approximately 124.66 mg. The amount of drug in Jane's body after 2 days is $A(2) = 100\left(1 + e^{1.4}\right)e^{-1.4(2)} \approx 30.741$, or approximately 30.74 mg. The amount of drug in Jane's body in the long run is given by $\lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left[100\left(1 + e^{1.4}\right)e^{-1.4t}\right] = 0$, or 0 mg.

b.



49. False. Take $a = b = x = 2$. Then the left-hand side is $(2 + 2)^2 = 16$, but the right-hand side is $2^2 + 2^2 = 8$.

50. True. $f(x) = e^x$ is an increasing function and so if $x < y$, then $f(x) < f(y)$, or $e^x < e^y$.

51. True. If $0 < b < 1$, then $f(x) = b^x$ is a decreasing function of x and so if $x < y$, then $f(x) > f(y)$; that is, $b^x > b^y$.

52. False. Take $k = x = -1$. Then $k < 0$ and $x < 0$, but $e^{kx} = e^1 > 1$.

53. True. If $k > 0$, then $f(x) = e^{kx} = \left(e^k\right)^x = b^x$ (where $b = e^k > 1$) and so f is increasing. If $k < 0$, then $f(x) = \left(e^k\right)^x = b^x$ (where $b = e^k < 1$) and so f is decreasing.

54. True. The functions $g(x) = x$ and $h(x) = 1 + e^x$ are both continuous on $(-\infty, \infty)$. Furthermore, $h(x) > 1 \neq 0$, so the quotient $f = g/h$ is continuous on $(-\infty, \infty)$.

Using Technology

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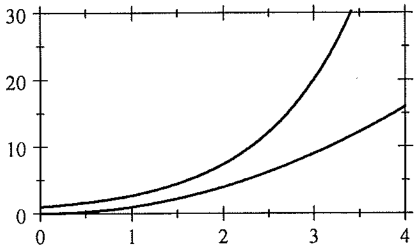

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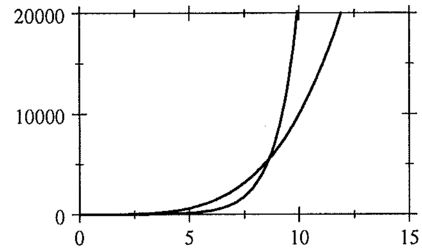

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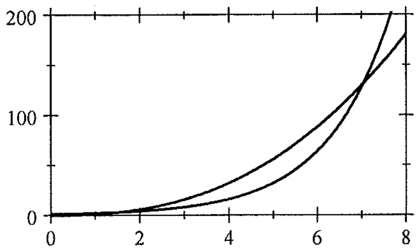

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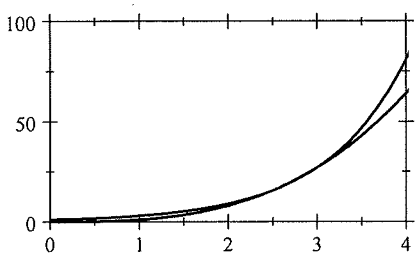

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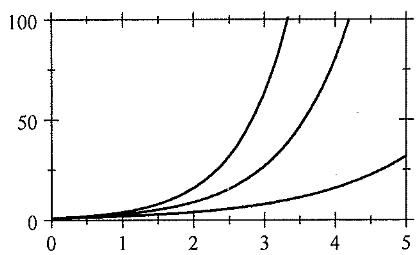

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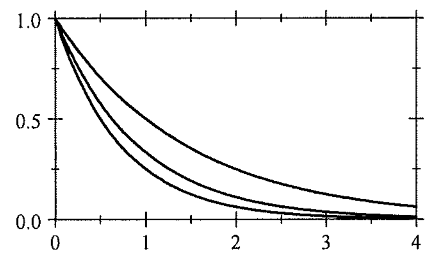

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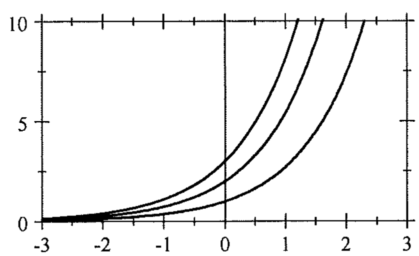

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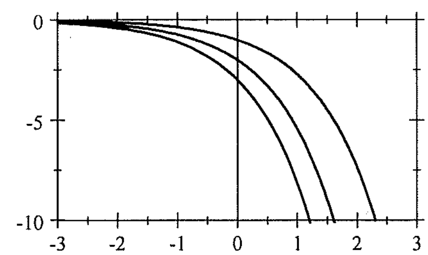

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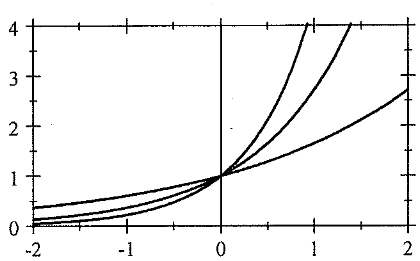

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